Journal of Sound and Vibration (1999) **226**(5), 831–871
Article No. jsvi.1999.2320, available online at http://www.idealibrary.com on IDEAL®

3-D THEORY VERSUS 2-D APPROXIMATE THEORY OF FREE ORTHOTROPIC (ISOTROPIC) PLATE AND SHELL VIBRATIONS, PART 2: NUMERICAL ALGORITHMS AND ANALYSIS

J. AWREJCEWICZ

Technical University of Lodz, Division of Automatics and Biomechanics, Stefanowskiego 1/15, 90-924 *Lodz, Poland*

AND

V. A. KRYSKO

Technical University of Saratov, Department of Mathematics, 41005 *Saratov, Russia*

(Received 15 June 1998, and in final form 1 March 1999)

The three-dimensional theory of orthotropic and isotropic plates (with and without concentrated masses) vibrations is used to estimate a validity of the two-dimensional theories application range. First, a general analytical approach is presented, and then the algorithms for numerical calculations are developed. Many examples obtained in the form of tables and drawings support the considerations and also some practically valid conclusions applied to isotropic and transversal-isotropic plates are derived.

1. INTRODUCTION

A great amount of literature has been devoted to the topic of the reduction of three-dimensional problems to the two-dimensional ones. Kiltchevski [1] has already pointed out that a general method for that reduction consists of constructing analytical expressions characterizing a stress–strain state by introduction of quantities defined in the $x0y$ co-ordinate system.

In reference [2] it has been shown that because of the lack of universal calculation models many different approximate methods have been applied.

An application range of the approximate theories is defined by a full three-dimensional theory. Therefore, the next logical step will include a comparison of the two-dimensional theory results with the three-dimensional ones. This approach can be used for qualitative estimation of the results of different, practically oriented theories. It allows for such a comparison because of the different characteristics such as displacements, frequencies and modes or vibration energy. It also seems that estimation of the errors which are introduced by the

0022-460X/99/400831 + 41 $30.00/0

two-dimensional theories, in comparison with those of the three-dimensional theory, is valid especially for a certain class of problems.

Now such a comparison will be outlined here on the basis of the following considerations: (1) free vibrations of an unloaded orthotropic (isotropic) plate; (2) free vibrations of an elastic system "orthotropic (isotropic) plate-concentrated masses".

2. DIFFERENTIAL EQUATIONS, BOUNDARY AND INITIAL CONDITIONS

From the expressions governing the displacement variations δu, δv, δw of the last equation of Part 1 of this paper, the following differential equations governing a stress-deformation dynamical state of the sloped shell are obtained:

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{13}}{\partial z} = \varrho \frac{\partial^2 u}{\partial t^2}, \quad \overleftrightarrow{(1,2)},$$

$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial y} + \frac{\partial \sigma_{33}}{\partial z} - k_1 \sigma_{11} - k_2 \sigma_{22} = \varrho \frac{\partial^2 w}{\partial t^2}. \tag{1}$$

Equations (1) can be replaced by an equivalent system. First one can disconnect the tension components. Then a vibration problem of a conical orthotropic shell with added elements is reduced to determination of the displacements components u, v, w satisfying the following equations:

$$A_{1111} \frac{\partial^2 u}{\partial x^2} + A_{1212} \frac{\partial^2 u}{\partial y^2} + A_{1313} \frac{\partial^2 u}{\partial z^2} + (A_{1122} + A_{1212}) \frac{\partial^2 v}{\partial x \partial y}$$

$$+ (A_{1133} + A_{1313}) \frac{\partial^2 w}{\partial x \partial z} (A_{1111} k_1 + A_{1122} k_2) \frac{\partial w}{\partial x} = \varrho \frac{\partial^2 u}{\partial t^2}, \quad \overleftrightarrow{(1,2)}$$

$$A_{3333} \frac{\partial^2 w}{\partial z^2} + A_{1313} \frac{\partial^2 w}{\partial x^2} + A_{2323} \frac{\partial^2 w}{\partial y^2} + (A_{1133} + A_{1313}) \frac{\partial^2 u}{\partial x \partial z}$$

$$+ (A_{2323} + A_{2233}) \frac{\partial^2 v}{\partial y \partial z} - 2(A_{1133} k_1 + A_{2233} k_2) \frac{\partial w}{\partial z} - (A_{1111} k_1$$

$$+ A_{2211} k_2) \frac{\partial u}{\partial x} - (A_{1122} k_1 + A_{2222} k_2) \frac{\partial v}{\partial y} + w(k_1^2 + k_2^2) = \varrho \frac{\partial^2 w}{\partial t^2}. \tag{2}$$

After introducing the dimensionless parameters

$$x = \bar{x} a,\ y = \bar{y} b,\ z = 2h\bar{z},\ u = 2h\bar{u},\ v = 2h\bar{v},\ w = 2h\bar{w},$$

$$\lambda_1 = a/2h,\ \lambda_2 = b/2h,$$

$$A_{1111} = A_{1111} \cdot \bar{A}_{1111}, \quad A_{1122} = \bar{A}_{1122} \cdot A_{1111}, \quad A_{1133} = A_{1111} \cdot \bar{A}_{1133},$$

$$A_{2222} = \bar{A}_{2222} \cdot A_{1111}, \quad A_{2233} = \bar{A}_{2233} \cdot A_{1111}, \quad A_{3333} = \bar{A}_{3333} \cdot A_{1111},$$

$$A_{1212} = A_{1111} \cdot \bar{A}_{1212}, \quad A_{1313} = A_{1111} \cdot \bar{A}_{1313}, \quad A_{2323} = A_{1111} \cdot \bar{A}_{2323},$$

$$t = \frac{ab}{2h}\sqrt{\varrho/A_{1111}} \cdot \bar{t}, \quad \omega = \frac{2h}{ab}\sqrt{A_{1111}/\varrho} \cdot \bar{\omega}, \quad \bar{M}^i = M^i/M_0;$$

$$J^i_{xx} = ab^3 2h\varrho \bar{J}^i_{xx}, \quad J^i_{yy} = a^3 b 2h\varrho \bar{J}^i_{yy}, \quad J^i_{xy} = a^2 b^2 2h\varrho \bar{J}^i_{xy},$$

$$J^i_{xz} = a^2 b (2h)^3 \varrho \bar{J}^i_{zz}, \quad J^i_{yz} = ab^2 (2h)^2 \varrho \bar{J}^i_{yz}, \quad J^i_{zz} = ab(2h)^3 \varrho \bar{J}^i_{zz},$$

$$k_1 = 2h/a^2 \bar{k}_1, \quad k_2 = 2h/b^2 \bar{k}_2, \quad \lambda = a/b, \tag{3}$$

the following dimensionless equations are obtained from equations (2) (bars are omitted):

$$A_{1111}\lambda_2^2 \frac{\partial^2 u}{\partial x^2} + A_{1212}\lambda_1^2 \frac{\partial^2 u}{\partial y^2} + A_{1313}\lambda_1^2\lambda_2^2 \frac{\partial^2 u}{\partial z^2} + (A_{1122} + A_{1212})\lambda_1\lambda_2 \frac{\partial^2 v}{\partial x \partial y}$$

$$+ (A_{1133} + A_{1313})\lambda_1\lambda_2^2 \frac{\partial^2 w}{\partial x \partial z} - (A_{1111}\lambda_2^2 k_1 + A_{1122}\lambda_1^2 k_2)\frac{1}{\lambda_1}\frac{\partial w}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \quad (\overleftrightarrow{1, 2}),$$

$$A_{3333}\lambda_1^2\lambda_2^2 \frac{\partial^2 w}{\partial z^2} + A_{1313}\lambda_2^2 \frac{\partial^2 w}{\partial x^2} + A_{2323}\lambda_1^2 \frac{\partial^2 w}{\partial y^2} + (A_{1133} + A_{1313})\lambda_1\lambda_2 \frac{\partial^2 u}{\partial x \partial y}$$

$$+ (A_{2323} + A_{2233})\lambda_1^2\lambda_2 \frac{\partial^2 v}{\partial y \partial z} - 2(A_{1133}\lambda_2^2 k_1 + A_{2233}\lambda_1^2 k_2)\frac{\partial w}{\partial x} - (A_{1111}\lambda_2^2 k_1$$
$$+ A_{2211}\lambda_1^2 k_2)\frac{1}{\lambda_1}\frac{\partial u}{\partial x} - (A_{1122}\lambda_2^2 k_1 + A_{2222}\lambda_1^2 k_2)\frac{\partial v}{\partial y} + w\left(\frac{k_1^2}{\lambda^2} + k_2^2\lambda^2\right) = \frac{\partial^2 w}{\partial t^2}. \tag{4}$$

The stiffness coefficients are

$$A_{1111} = (a_{2222}a_{3333} - a_{2233}a_{3322})/\Delta, \quad (\overleftrightarrow{1, 2, 3})$$

$$A_{1122} = (a_{1133}a_{3322} - a_{1122}a_{3333})/\Delta, \quad (\overleftrightarrow{1, 2, 3})$$

$$A_{1212} = \frac{1}{a_{1212}}, \quad A_{1313} = \frac{1}{a_{1313}}, \quad A_{2323} = \frac{1}{a_{2323}},$$

$$A_{2211} = A_{1122}, \quad (\overleftrightarrow{1, 2, 3}), \quad \Delta = \det[a_{iijj}]_{i,j=1,\dots,3}, \tag{5}$$

and additionally the following relations are valid:

$$a_{1111} = \frac{1}{E_1}, \quad a_{1122} = -\frac{\nu_{12}}{E_2}, \quad a_{1133} = -\frac{\nu_{13}}{E_3}, \quad a_{2211} = -\frac{\nu_{21}}{E_1},$$

$$a_{2222} = \frac{1}{E_2}, \quad a_{2233} = -\frac{\nu_{23}}{E_3}, \quad a_{3311} = -\frac{\nu_{31}}{E_1}, \quad a_{3322} = -\frac{\nu_{32}}{E_2},$$

$$a_{3333} = \frac{1}{E_3}, \quad a_{1313} = \frac{1}{G_{13}}, \quad a_{2323} = \frac{1}{G_{23}}, \quad a_{1212} = \frac{1}{G_{12}}. \tag{6}$$

E_1, E_2, E_3 are Young's moduli ($E_2\nu_{21} = E_1\nu_{12}$, $E_3\nu_{32} = E_2\nu_{23}$, $E_1\nu_{13} = E_3\nu_{31}$; for the isotropic material $E_1 = E_2 = E$); ν_{ij} are the Poisson's coefficients (for the isotropic material $\nu_{12} = \nu_{21} = \nu$); G_{12}, G_{13}, G_{23} are the shear moduli (for the isotropic material $G = G_{12}$ is the shear modulus for the planes parallel to the isotropic plane, and $G' = G_{13} = G_{22}$ is the shear modulus for the planes normal to the plane of isotropy); x, y, z are the Cartesian co-ordinates; u, v, w are the displacements of the mean surface in the x, y, z directions, respectively; a, b are the dimension of a plate; k_1, k_2 are the curvatures of the plate; and t is the time.

A solution to the differential equations (4) will be constructed by using a condition allowing for a slip on the plate's edges and assuming the absence of additional masses on its external surface.

This means that the following boundary conditions are used: $\sigma_{11} = 0$, $\upsilon = 0$, $w = 0$.

The displacement functions which fulfil the above-mentioned boundary conditions are as follows:

$$u = \sum_{m,n}^{\infty} \phi(z) \cos \alpha_m x \sin \beta_n y \sin \omega_{mn} t,$$

$$v = \sum_{m,n}^{\infty} \psi(z) \sin \alpha_m x \cos \beta_n y \sin \omega_{mn} t,$$

$$w = \sum_{m,n}^{\infty} \chi(z) \sin \alpha_m x \sin \beta_n y \sin \omega_{mn} t, \tag{7}$$

where $\alpha_m = m\pi$, $\beta_n = n\pi$, m, n are integers, and ω_{mn} are the free vibration frequencies. Substituting equations (7) into equations (4) with $k_1 = k_2 = 0$ yields the following uncoupled three ordinary differential equations for different m, n, which must be satisfied by the functions ϕ, ψ, and χ:

$$\xi_{11} \frac{d^2\phi}{dz^2} = \xi_{12} \frac{d\chi}{dz} + \xi_{13}\phi + \xi_{14}\psi,$$

$$\xi_{21}\frac{d^2\psi}{dz^2} = \xi_{22}\frac{d\chi}{dz} + \xi_{23}\phi + \xi_{24}\psi,$$

$$\xi_{31}\frac{d^2\chi}{dz^2} = \xi_{32}\frac{d\phi}{dz} + \xi_{33}\frac{d\psi}{dz} + \xi_{34}\chi. \tag{8}$$

Here ξ_{ij} are functions of m, n and of the physical and geometrical characteristics of plates of the form

$$\xi_{11} = A_{1313}\lambda_1^2\lambda_2^2, \qquad \xi_{12} = -(A_{1133} + A_{1313})\lambda_1\lambda_2^2\alpha_m,$$

$$\xi_{13} = A_{1111}\lambda_2^2\alpha_m^2 + A_{1212}\lambda_1^2\beta_n^2 - \omega_{mn}^2, \qquad \xi_{14} = (A_{1122} + A_{1212})\lambda_1\lambda_2\alpha_m\beta_n,$$

$$\xi_{21} = A_{2323}\lambda_1^2\lambda_2^2, \qquad \xi_{22} = -(A_{2233} + A_{2323})\lambda_1^2\lambda_2\beta_n,$$

$$\xi_{23} = (A_{1122} + A_{1212})\lambda_1\lambda_2\alpha_m\beta_n, \qquad \xi_{24} = A_{2222}\lambda_1^2\beta_n^2 + A_{1212}\lambda_2^2\alpha_m^2 - \omega_{mn}^2,$$

$$\xi_{31} = A_{3333}\lambda_1^2\lambda_2^2, \qquad \xi_{32} = (A_{1313} + A_{1133})\lambda_1\lambda_2^2\alpha_m,$$

$$\xi_{33} = (A_{2233} + A_{2323})\lambda_1^2\lambda_2\beta_n, \quad \xi_{34} = A_{1313}\lambda_2^2\alpha_m^2 + A_{2323}\lambda_1^2\beta_n^2 - \omega_{mn}^2. \tag{9}$$

The solutions of equations (8) are sought in the form

$$\{\phi, \psi, \chi) = \{A, B, C\}\mathrm{e}^{\tau z}, \tag{10}$$

where the parameter τ is obtained from the equation

$$r_1\tau^6 + r_2\tau^4 + r_3\tau^2 + r_4 = 0. \tag{11}$$

with each of the six roots of equation (11) is associated a particular solution is denoted as η_i. Therefore, the general solution has the form

$$[\eta] = \sum_{i=1}^{6} [\eta_i]. \tag{12}$$

A procedure for solution of equation (12) has some difficulties because it is impossible to find the solutions of equation (11) for each orthotropic material. Therefore, at this stage one needs to assume the existence of different combinations of roots (without the conjugated roots). Theoretically, the following roots combinations are possible: (1) simple; (2) one simple and one multiple; (3) all equal. One can now consider the solution construction for each of the mentioned cases [3].

1. *Simple roots of equation* (11). Substitute the solution (10) into the differential equation system (8). Division by an exponential multiplier yields

$$LR = 0, \tag{13}$$

where

$$L = \begin{bmatrix} \xi_{13} - \xi_{11}\tau^2 & \xi_{14} & \xi_{12}\tau \\ \xi_{23} & \xi_{14} - \xi_{21}\tau^2 & \xi_{22}\tau \\ \xi_{32}\tau & \xi_{33}\tau & \xi_{34} - \xi_{31}\tau^2 \end{bmatrix}, \quad R = \begin{bmatrix} A \\ B \\ C \end{bmatrix}. \tag{14}$$

The determinant with the unknown A, B, C coefficients serves to find the roots. The rank of the coefficients' matrix has to be equal to two and then there exists at least one minor of the second order. From the three linearly dependent equations one can take those two which possess coefficients dependent on the minor. From the two linear equations obtained one can define two constants as a function of the third one. For instance, if the second order minor constructed by the coefficients denoted by A and B in the last two equations of the system analyzed is different from zero, then the following vector of the fundamental solutions is defined:

$$[\eta'] = C \begin{bmatrix} \dfrac{-\xi_{22}\xi_{33}\tau^2 - (\xi_{34} - \xi_{31}\tau^2)(\xi_{31}\tau^2 - \xi_{24})}{\xi_{23}\xi_{33}\tau + \xi_{32}\tau(\xi_{21}\tau^2 - \xi_{34})} \\ \dfrac{\xi_{23}(\xi_{31}\tau^2 - \xi_{34}) + \xi_{32}\xi_{22}\tau}{\xi_{23}\xi_{33}\tau + \xi_{32}\tau(\xi_{21}\tau^2 - \xi_{34})} \\ 1 \end{bmatrix} e^{\tau z}. \tag{15}$$

2. *Multiple roots of equation* (11). When $\tau = \tau^x$ is a multiple root then the corresponding solution (10) is substituted into equation (8). Transforming the algebraic system (13) with the unknown A, B, C yields a rank which is equal to or greater than 1. Suppose that the system coefficients matrix rank is equal to one. In that case, the analyzed system is equivalent to an arbitrarily taken one from two equations, in which at least one coefficient is different from zero. This leads to the conclusion that to define three unknown constants one has one relationship. Therefore, two constants can be chosen optionally and the fundamental system possesses two vectors. For example, using the third equation of (14) yields

$$-A\xi_{32}\tau - B\xi_{33}\tau + C(\xi_{31}\tau^2 - \xi_{34}) = 0. \tag{16}$$

Thus, the eigenvector may be defined by

$$[\eta'] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{(\xi_{31}\tau^2 - \xi_{34})}{\xi_{22}\tau} & \dfrac{(\xi_{31}\tau^2 - \xi_{34})}{\xi_{34}\tau} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{\tau^x z}. \tag{17}$$

Suppose that τ^x is of second power. Then the vector components have the form

$$[\eta'] = \begin{bmatrix} A_1 + B_1 z \\ A_2 + B_2 z \\ A_3 + B_3 z \end{bmatrix} e^{\tau^x z}. \tag{18}$$

Substituting the relationship (18) into the differential equations and comparing the coefficients of the same powers yields six linear equations. The rank of the obtained $[6 \times 6]$ matrix is equal to four and therefore one has four equations to define six unknowns. Taking two unknowns arbitrarily and joining with them either row or column values of an arbitrarily taken second order and non-zero determinant, yields the fundamental solution system which consists of two vectors with six components.

To conclude, it is possible in the case analyzed to construct the solutions to the differential equations (8).

3. *Triple root.* One takes the solution (18) and substitutes it into the system (8). Comparing the coefficients with the same "z" powers yields six linear equations. The matrix rank is greater than or equal to three. Proceeding as in the above case, one can also construct a general solution. We do not focus on a detailed analysis here but we have to emphasize that this case is practically realized during the analysis of a three-dimensional isotropic plate deflection subjected to an external surface load action [4].

The detailed calculations have shown, that for an isotropic plate material one has real and different roots of the characteristic equation (11). A matrix rank of the linear system (13) is equal to two. The minors constructed of coefficients represented by the unknowns in the last two algebraic equations are different from zero. Suppose that equation (11) has the roots τ_1, τ_2, τ_3. A general solution of equation (8) may be obtained as a linear combination of the particular solutions (15)

$$\begin{aligned} \phi(z) &= Ad_{11}^{(1)} \operatorname{ch} \tau_1 z + Bd_{11}^{(1)} \operatorname{sh} \tau_1 z + Cd_{11}^{(2)} \operatorname{ch} \tau_2 z \\ &\quad + Dd_{11}^{(2)} \operatorname{sh} \tau_2 z + Ed_{11}^{(3)} \operatorname{ch} \tau_3 z + Fd_{11}^{(3)} \operatorname{sh} \tau_3 z, \\ \psi(z) &= Ad_{12}^{(1)} \operatorname{ch} \tau_1 z + Bd_{12}^{(1)} \operatorname{sh} \tau_1 z + Cd_{12}^{(2)} \operatorname{ch} \tau_2 z \\ &\quad + Dd_{12}^{(2)} \operatorname{sh} \tau_2 z + Ed_{12}^{(3)} \operatorname{ch} \tau_3 z + Fd_{12}^{(3)} \operatorname{sh} \tau_3 z, \\ \chi(z) &= Ad_{13}^{(1)} \operatorname{sh} \tau_1 z + Bd_{13}^{(1)} \operatorname{ch} \tau_1 z + Cd_{13}^{(2)} \operatorname{sh} \tau_2 z \\ &\quad + Dd_{13}^{(2)} \operatorname{ch} \tau_2 z + Ed_{13}^{(3)} \operatorname{sh} \tau_3 z + Fd_{13}^{(3)} \operatorname{ch} \tau_3 z, \end{aligned} \tag{19}$$

where

$$\begin{aligned} d_{11}^{(i)} &= (\xi_{34} + \tau_i^2 \xi_{21})(\xi_{34} + \xi_{31}\tau_i^2) + \xi_{22}\xi_{33}\tau_i^2, \\ d_{12}^{(i)} &= -\xi_{32}\xi_{22}\tau_i^2 - \xi_{23}(\xi_{34} - \xi_{31}\tau_i^2), \\ d_{13}^{(i)} &= \tau_i[\xi_{23}\xi_{33} - \xi_{32}(\xi_{24} + \tau_i^2 \xi_{21})], \quad i = 1, \ldots, 3. \end{aligned} \tag{20}$$

In the free vibration case, the external boundary conditions are formulated by using the relations

$$\sigma_{13} = 0, \quad \sigma_{23} = 0, \quad \sigma_{33} = 0. \tag{21}$$

They have the following form (with respect to ϕ, ψ, χ and for $z = -0{\cdot}5, 0{\cdot}5$):

$$-A_{1133}\tilde{\alpha}_m\phi - A_{2233}\tilde{\beta}_n\psi + A_{3333}\chi' = 0,$$

$$\psi' + \tilde{\beta}_n\chi = 0, \quad \phi' + \tilde{\alpha}_m\chi = 0, \quad \tilde{\alpha}_m = \alpha_m/\lambda, \quad \tilde{\beta}_n = \beta_n/\lambda_2. \tag{22}$$

Substituting expressions for ϕ, ψ, χ from equations (19) into equations (22) one obtains the following system of homogeneous algebraic equations for each pair of (m, n):

$$\begin{bmatrix} [\mathrm{L}(0,5)] \\ [\mathrm{L}(-0,5)] \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{23}$$

Here

$$L(z) = \begin{bmatrix} l_1(\tau_1)\,\mathrm{ch}\,\tau_1 z & l_1(\tau_1)\,\mathrm{sh}\,\tau_1 z & \cdots & l_1(\tau_3)\,\mathrm{ch}\,\tau_3 z & l_1(\tau_3)\,\mathrm{sh}\,\tau_3 z \\ l_2(\tau_1)\,\mathrm{sh}\,\tau_1 z & l_2(\tau_1)\,\mathrm{ch}\,\tau_1 z & \cdots & l_2(\tau_3)\,\mathrm{sh}\,\tau_3 z & l_2(\tau_3)\,\mathrm{ch}\,\tau_3 z \\ l_3(\tau_1)\,\mathrm{sh}\,\tau_1 z & l_3(\tau_1)\,\mathrm{ch}\,\tau_1 z & \cdots & l_3(\tau_3)\,\mathrm{sh}\,\tau_3 z & l_3(\tau_3)\,\mathrm{ch}\,\tau_3 z \end{bmatrix},$$

$$l_1(\tau_i) = A_{1133}\tilde{\alpha}_m d_{11}^{(i)}(\tau_i) + A_{2233}\tilde{\beta}_n d_{12}^{(i)}(\tau_i) + A_{3333}\, d_{13}^{(i)}(\tau_i)\tau_i,$$

$$l_2(\tau_i) = d_{11}^{(i)}(\tau_i)\tau_i + d_{13}^{(i)}(\tau_i)\tilde{\alpha}_m,$$

$$d_{11}^{(i)} = (\xi_{24} - \tau_i^2\xi_{21})(\xi_{34} - \xi_{31}\tau_i^2) - \xi_{22}\xi_{33}\tau_i^2,$$

$$d_{12}^{(i)} = \xi_{32}\xi_{22}\tau_i^2 - \xi_{23}(\xi_{34} - \xi_{31}\tau_i^2),$$

$$d_{13}^{(i)} = \tau_i[\xi_{23}\xi_{33} - \xi_{32}(\xi_{24} - \tau_i^2\xi_{21})],$$

$$l_3(\tau_i) = d_{11}^{(i)}(\tau_i)\tau_i + d_{13}^{(i)}(\tau_i)\tilde{\alpha}_m, \quad (i = 1, \ldots, 3). \tag{24}$$

In order to obtain non-trivial solutions one assumes

$$\det\begin{bmatrix} [\mathrm{L}(0,5)] \\ [\mathrm{L}(-0,5)] \end{bmatrix} = 0. \tag{25}$$

As a result, one has a transcendental equation whose roots correspond to the transversally isotropic plate vibration frequencies. For each pair (m, n), one obtains an infinite set of eigenvalues.

The free vibrations analysis of a cuboid made from the isotropic material is of specific importance. In that case general considerations are simplified and lead to relatively simple characteristics equations. The particular solutions (modes) of isotropic plates vibrations are to be investigated. They are needed for both the free and excited vibrations as well as for non-stationary waves investigations. For this reason, these solutions have attracted many researchers and they have been used mainly for non-stationary processes analysis [5]. In the case of a harmonically vibrated isotropic plate with the 'ω', frequency the fundamental solutions are simplified to the following:

$$\frac{1}{\lambda_2^2}\frac{\partial^2 u}{\partial x^2}+\frac{1}{\lambda_2^2}\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{(1-2\nu)}\left(\frac{1}{\overline{\lambda}_2^2}\frac{\partial^2 u}{\partial x^2}+\frac{1}{\lambda_1\lambda_2}\frac{\partial^2 v}{\partial x\partial y}\right.$$

$$\left.+\frac{1}{\lambda_1}\frac{\partial^2 w}{\partial x\partial z}\right)+u\omega^2=0, \quad (\overleftrightarrow{1,2}), \quad (\overleftrightarrow{x,y}),$$

$$\frac{1}{\lambda_1^2}\frac{\partial^2 w}{\partial x^2}+\frac{1}{\lambda_2^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{(1-2\nu)}\left(\frac{1}{\lambda_1}\frac{\partial^2 u}{\partial x\partial y}+\frac{1}{\lambda_2}\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}\right)+w\omega^2=0. \quad (26)$$

The fundamental functions (7) fulfilling the boundary conditions (22) are taken, and by carrying out a procedure analogous to that described in reference [3], the following characteristic equation is found:

$$\{8g^2 rs(r^2+g^2)^2(1-\operatorname{ch} r \operatorname{ch} s)+[16g^4r^2s^2+(r^2+g^2)^4]\operatorname{sh} r \operatorname{sh} s)\operatorname{sh} r=0,$$

$$g=\sqrt{\tilde{\alpha}_m^2+\tilde{\beta}_n^2}, \quad r=\sqrt{g^2-\omega^2}, \quad s=\sqrt{g^2-(1-2\nu)\omega^2/(2-2\nu)}. \quad (27)$$

The solution of this transcendental equation attaches infinite eigenvalues series to each pair (m, n). In reference [3] the authors conclude that equation (25) contains the eigenvalues corresponding to symmetric and antisymmetric plate vibrations in relation to a mean surface (in reference [6] it has been shown that such a distribution is always possible).

Another challenging method has been presented in reference [7], where a general solution is described for two classes, and their particular solutions correspond to plane and antiplane vibrations.

Earlier, in reference [8], the free vibrations of a simply supported rectangular plate using the three-dimensional theory, have been analyzed. It has been shown that from the characteristic equation the roots, corresponding to the modified (more accurate) differential equations governing a plate vibration, can be found.

3. THE 3-D PROBLEM (WITH THE ADDED MASSES)

In this section, the orthotropic plate vibrations with the added masses M^i $(i = 1, \dots, N)$ will be analyzed. It should be emphasized that when using the three-dimensional theory for an orthotropic plate in a general case, one does not have prior knowledge of the behaviour of the roots of equation (11), because unknown frequency is included there. In addition, coefficients are defined by complex expressions dependent on the physical and geometrical parameters, and even an analytical description for the roots does not simplify the problem. The classification of possible roots types of the characteristic equation as well as the method of their corresponding solution should be outlined. One assumes here, the following section 2, that physical and geometrical parameters of the plate allow one to obtain three simple roots of equation (11), and a corresponding matrix of the linear equations (13) has the rank equal to 2.

The following solution to the equations (4) is being sought (one also takes $k_1 = k_2 = 0$ in equations (4)),

$$u = \sin \omega t \sum_{m,n}^{\infty} \phi(z) \cos \alpha_m x \sin \beta_n y,$$

$$v = \sin \omega t \sum_{m,n}^{\infty} \psi(z) \sin \alpha_m x \cos \beta_n y,$$

$$w = \sin \omega t \sum_{m,n}^{\infty} \chi(z) \sin \alpha_m x \sin \beta_n y. \tag{28}$$

Here ϕ, ψ and χ are defined by equation (19).

In order to find A_{mn}, B_{mn}, C_{mn}, D_{mn},E_{mn}, F_{mn} one substitutes equation (28) into the boundary conditions on the surfaces $z = -0{\cdot}5$, $0{\cdot}5$, which for $z = 0{\cdot}5$ have the following non-dimensional form:

$$A_{1313}\left(\frac{\partial U}{\partial z} + \frac{1}{\lambda_1}\frac{\partial W}{\partial x}\right) = \omega^2 \sum_{j=1}^{N} M^i \left[U - \frac{1}{\lambda_1}(z_i - 0{\cdot}5)\frac{\partial W}{\partial x}\right] \delta(x - x_i) x_i$$
$$\times (\delta y - y_i), \quad (\overleftrightarrow{1, 2}), \quad (\overleftrightarrow{x, y}),$$

$$A_{1133}\frac{1}{\lambda_1}\frac{\partial U}{\partial x} + A_{2233}\frac{1}{\lambda_2}\frac{\partial V}{\partial y} + A_{3333}\frac{\partial W}{\partial z}$$

$$= \omega^2 \sum_{i=1}^{N} \left\{ M^i \left[W - \frac{(z_i - 0{\cdot}5)^2}{\lambda_1^2}\frac{\partial}{\partial x}\left[\frac{\partial W}{\partial x}\delta(x - x_i)\right]\delta(y - y_i) + \frac{(z_i - 0{\cdot}5)}{\lambda_1}\right.\right.$$

$$\times \frac{\partial}{\partial x}[U\delta(x - x_i)]\,\delta(y - y_i) - \frac{(z_i - 0{\cdot}5)^2}{\lambda_2^2}\frac{\partial}{\partial y}\left[\frac{\partial W}{\partial y}\delta(y - y_i)\right]\delta(x - x_i)$$

$$+\frac{(z_i-0{\cdot}5)}{\lambda_2}\frac{\partial}{\partial y}[\delta(y-y_i)]\,\delta(x-x_i)\Bigg]\Bigg\}-\sum_{i=1}^{N}\left\{J^i_{xx}\frac{\partial}{\partial y}\left[\frac{\partial W}{\partial y}\delta(y-y_i)\right]\delta(x-x_i)\right.$$

$$\left.-J^i_{yy}\frac{\partial}{\partial x}\left[\frac{\partial W}{\partial x}\delta(x-x_i)\right]\delta(y-y_i)\right\}, \tag{29}$$

and for $z=-0{\cdot}5$ one gets

$$\frac{\partial U}{\partial z}+\frac{1}{\lambda_1}\frac{\partial W}{\partial x}=0,\quad \overleftrightarrow{(1,2)},\quad \overleftrightarrow{(x,y)},$$

$$A_{1133}\frac{1}{\lambda_1}\frac{\partial U}{\partial x}+A_{2233}\frac{1}{\lambda_2}\frac{\partial V}{\partial y}+A_{3333}\frac{\partial W}{\partial z}=0, \tag{30}$$

where U, V, W are the amplitudes of the displacements:

$$u=U\sin\omega t,\quad v=V\sin\omega t,\quad w=W\sin\omega t.$$

Substituting equations (28) into equations (29) and (30) and developing the right-hand sides of equation (29) with the singular coefficients of the δ-type into trigonometric series yields the following linear algebraic equations:

$$LA=Q, \tag{31}$$

where

$$L=\begin{bmatrix} l_{11} & l_{12} & \cdots & l_{16}\\ l_{21} & l_{22} & \cdots & l_{26}\\ \cdots & \cdots & \cdots & \cdots\\ l_{61} & l_{62} & \cdots & l_{66}\end{bmatrix},\quad A=\begin{bmatrix}A_{mn}\\ B_{mn}\\ C_{mn}\\ D_{mn}\\ E_{mn}\\ F_{mn}\end{bmatrix},\quad Q=\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ 0\\ 0\\ 0\end{bmatrix}.$$

The L-matrix coefficients and the Q rows have the following forms

$$l_{1k}=p_j\varphi_k^+,\quad l_{1m}=p_j\varphi_m^+,\quad l_{2k}=r_j\varphi_k^+,\quad l_{2m}=r_j\varphi_m^+,$$

$$l_{3k}=s_j\varphi_m^+,\quad l_{3m}=s_j\varphi_k^+,\quad l_{4k}=p_j\varphi_k^-,\quad l_{4m}=p_j\varphi_m^-,$$

$$l_{5k}=r_j\varphi_k^-,\quad l_{5m}=r_j\varphi_m^-,\quad l_{6k}=s_j\varphi_m^-,\quad l_{6m}=s_j\varphi_k^-,$$

$$\varphi_k^{\pm}=sh(\pm 0,5\tau_j),\quad \varphi_m^{\pm}=\mathrm{ch}(\pm 0,5\tau_j),$$

$$p_j = A_{1313}(\tau_j d_{11}^{(j)} + \tilde{\alpha}_m d_{13}^{(j)}), \quad r_j = A_{2323}(\tau_j d_{12}^{(j)} + \tilde{\beta}_n d_{13}^{(j)}),$$

$$s_j = - A_{1133}\tilde{\alpha}_m d_{11}^{(j)} - A_{2233}\tilde{\beta}_n d_{12}^{(j)} + A_{3333}\tau_j d_{13}^{(j)},$$

$$\tilde{\alpha}_m = \alpha_m/\lambda, \quad \tilde{\beta}_n = \beta_n/\lambda_2, \quad (k = 2j-1,\, m = 2j,\, j = 1, \dots, 3).$$

$$\xi_1 = \omega^2 \sum_{i=1}^{N} \left(g_{11}^i U_i + g_{12}^i \frac{\partial W_i}{\partial x} \right), \quad \xi_2 = \omega^2 \sum_{i=1}^{N} \left(g_{21}^i \mathrm{V}_i + \mathrm{g}_{22}^i \frac{\partial W_i}{\partial y} \right),$$

$$\xi_3 = \omega^2 \sum_{i=1}^{N} \left(g_{31}^i U_i + g_{32}^i V_i + g_{33}^i W_i + g_{34}^i \frac{\partial W_i}{\partial x} + g_{35}^i \frac{\partial W_i}{\partial y} \right). \tag{32}$$

The parameters g_{ij}^k are defined by

$$g_{11}^i = 4M^i \cos\alpha_m x_i \sin\beta_n y_i, \quad g_{12}^i = -\frac{4M^i}{\lambda_1}(z_i - 0{\cdot}5)\cos\alpha_m x_i \sin\beta_n y_i,$$

$$g_{21}^i = 4M^i \sin\alpha_m x_i \cos\beta_n y_i, \quad g_{22}^i = -\frac{4M^i}{\lambda_2}(z_i - 0{\cdot}5)\sin\alpha_m x_i \cos\beta_n y_i,$$

$$g_{31}^i = -\frac{4M^i}{\lambda_1}(z_i - 0{\cdot}5)\,\alpha_m \cos\alpha_m x_i \sin\beta_n y_i,$$

$$g_{32}^i = \frac{4M^i}{\lambda_1^2}(z_i - 0{\cdot}5)\sin\alpha_m x_i \cos\beta_n y_i, \quad g_{33}^i = 4M^i \sin\alpha_m x_i \sin\beta_n y_i,$$

$$g_{34}^i = \frac{4M^i}{\lambda_1^2}\alpha_m \cos\alpha_m x_i \sin\beta_n y_i + 4J_{xx}^i \alpha_m \cos\alpha_m x_i \sin\beta_n y_i,$$

$$g_{35}^i = \frac{4M^i}{\lambda_2^2}\beta_n \sin\alpha_m x_i \cos\beta_n y_i + 4J_{xx}^i \beta_n \sin\alpha_m x_i \cos\beta_n y_i, \quad (i = 1, \dots, N). \tag{33}$$

Finding the solutions of equations (31) one defines A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}, F_{mn}, which implies a definition of the displacement component vector:

$$U = \omega^2 \sum_{m,n}^{\infty} \sum_{i=1}^{N} \frac{1}{\Delta(m,n)} \left(q_{11}^i U_i + q_{12}^i V_i + q_{13}^i W_i + q_{14}^i \frac{\partial W_i}{\partial x} + q_{15}^i \frac{\partial W_i}{\partial y} \right) \cos\alpha_m x \sin\beta_n y,$$

$$V = \omega^2 \sum_{m,n}^{\infty} \sum_{i=1}^{N} \frac{1}{\Delta(m,n)} \left(q_{21}^i U_i + q_{22}^i V_i + q_{23}^i W_i + q_{24}^i \frac{\partial W_i}{\partial x} + q_{25}^i \frac{\partial W_i}{\partial y} \right) \sin\alpha_m x \cos\beta_n y,$$

$$W = \omega^2 \sum_{m,n}^{\infty} \sum_{i=1}^{N} \frac{1}{\Delta(m,n)} \left(q_{31}^i U_i + q_{32}^i V_i + q_{33}^i W_i + q_{34}^i \frac{\partial W_i}{\partial x} + q_{35}^i \frac{\partial W_i}{\partial y} \right) \sin\alpha_m x \sin\beta_n y. \tag{34}$$

The plate's first deflection derivatives are

$$\frac{\partial W}{\partial x} = \omega^2 \sum_{m,n}^{\infty} \sum_{i=1}^{N} \frac{1}{\Delta(m,n)} \left(q_{31}^i U_i + q_{32}^i V_i + q_{33}^i W_i + q_{34}^i \frac{\partial W_i}{\partial x} + q_{35}^i \frac{\partial W_i}{\partial y} \right)$$

$$\times \alpha_m \cos \alpha_m x \sin \beta_n y,$$

$$\frac{\partial W}{\partial y} = \omega^2 \sum_{m,n}^{\infty} \sum_{i=1}^{N} \frac{1}{\Delta(m,n)} \left(q_{31}^i U_i + q_{32}^i V_i + q_{33}^i W_i + q_{34}^i \frac{\partial W_i}{\partial x} + q_{35}^i \frac{\partial W_i}{\partial y} \right)$$

$$\times \beta_n \sin \alpha_m x \cos \beta_n y, \tag{35}$$

$$q_{jk}^i = p_{1k}^i d_{1j}^{(1)} \operatorname{ch} 0{\cdot}5\tau_1 + p_{2k}^i d_{1j}^{(1)} \operatorname{sh} 0{\cdot}5\tau_1 + \cdots + p_{5k}^i d_{1j}^{(3)} \operatorname{ch} 0{\cdot}5\tau_3 + p_{6k}^i d_{1j}^{(3)} \operatorname{sh} 0{\cdot}5\tau_3,$$
$$(j = 1, 2, \quad k = 1, \ldots, 5),$$

$$q_{3k}^i = p_{1k}^i d_{13}^{(1)} \operatorname{sh} 0{\cdot}5\tau_1 + p_{2k}^i d_{13}^{(1)} \operatorname{ch} 0{\cdot}5\tau_1 + \cdots + p_{5k}^i d_{13}^{(3)} \operatorname{sh} 0{\cdot}5\tau_3 + p_{6k}^i d_{13}^{(3)} \operatorname{ch} 0{\cdot}5\tau_3,$$

$$p_{j1}^i = g_{11}^i \Delta_{1j} + g_{31}^i \Delta_{3j}, \quad p_{j2}^i = g_{21}^i \Delta_{2j} + g_{33}^i \Delta_{3j}, \quad p_{j3}^i = \Delta_{3j} g_{33}^i,$$

$$p_{j4}^i = g_{12}^i \Delta_{1j} + g_{31}^i \Delta_{3j}, \quad p_{j5}^i = g_{22}^i \Delta_{2j} + g_{35}^i \Delta_{3j}, \quad (j = 1, \ldots, 6;\ i = 1, \ldots, N). \tag{36}$$

Δ is the determinant of equation (31), whereas Δ_{ij} is the co-factor of a term standing on the cross-point of ith row and jth column.

After the fulfilment of the continuous conditions of the jointed masses, one obtains the algebraic linear equations with the unknowns being the displacement vector components and the first deflection derivatives in the point of the joined masses,

$$U_i,\ V_i,\ W_i,\ \partial W_i/\partial x,\ \partial W_i/\partial y \tag{37}$$

A non-trivial solution exists if

$$\det[a_{ij}]_{5N \times 5N} = 0. \tag{38}$$

The determinant elements are given in Table 1. The solution of equation (38) does not exist in a closed form. It is a transcendental equation, because the frequency sought is the hyperbolic function argument. Limiting the considerations to the first term yields an approximate equation defining the frequency of the vibration "plate–mass" system. This equation has infinitely many solutions which are the approximate frequency values of the free vibrations. Taking into account two terms in equation (38) yields more accurate results for the frequency determination and one can define a new frequency series corresponding to the second vibration mode, and so on. Limiting oneself to "s" terms in equation (38) one can achieve the required accuracy in a frequency determination.

TABLE 1

Elements of determinant

$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^1 \lambda_1^1}{\Delta(m,n)} - 1$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{12}^1 \lambda_1^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{15}^1 \lambda_1^1}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^2 \lambda_1^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{15}^2 \lambda_1^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^N \lambda_1^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^N \lambda_1^1}{\Delta(m,n)}$
$\omega^2 \sum_{m,n}^{\infty} \frac{q_{21}^1 \lambda_1^2}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{22}^1 \lambda_1^2}{\Delta(m,n)} - 1 \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{25}^1 \lambda_1^2}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{21}^2 \lambda_1^2}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{25}^2 \lambda_1^2}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{21}^N \lambda_1^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{25}^N \lambda_2^1}{\Delta(m,n)}$
$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^1 \lambda_1^5}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{32}^1 \lambda_1^5}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{35}^1 \lambda_1^5}{\Delta(m,n)} - 1$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^2 \lambda_1^5}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{35}^N \lambda_1^5}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{35}^N \lambda_1^5}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{35}^N \lambda_1^5}{\Delta(m,n)}$
$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^1 \lambda_2^1}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{12}^1 \lambda_2^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{15}^1 \lambda_2^1}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^2 \lambda_2^1}{\Delta(m,n)} - 1 \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{15}^2 \lambda_2^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^N \lambda_2^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^N \lambda_2^1}{\Delta(m,n)}$
$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^1 \lambda_2^5}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{32}^1 \lambda_1^5}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{35}^1 \lambda_2^5}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^2 \lambda_2^5}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{35}^2 \lambda_2^5}{\Delta(m,n)} - 1 \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^N \lambda_2^5}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^N \lambda_2^5}{\Delta(m,n)}$
$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^1 \lambda_N^1}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{12}^1 \lambda_N^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{15}^1 \lambda_N^1}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^2 \lambda_N^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{15}^2 \lambda_N^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^N \lambda_N^1}{\Delta(m,n)} - 1 \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{11}^N \lambda_N^1}{\Delta(m,n)}$
$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^1 \lambda_N^1}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{32}^1 \lambda_N^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{35}^1 \lambda_N^1}{\Delta(m,n)}$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^2 \lambda_N^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{35}^2 \lambda_N^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{q_{31}^N \lambda_N^1}{\Delta(m,n)} \cdots$	$\omega^2 \sum_{m,n}^{\infty} \frac{p_{31}^N \lambda_N^1}{\Delta(m,n)} - 1$

$$\lambda_i^1 = \cos\alpha_m x_i \sin\beta_n y_i,\ \lambda_i^2 = \sin\alpha_m x_i \cos\beta_n y_i,\ \lambda_i^3 = \sin\alpha_m x_i \sin\beta_n y_i,$$
$$\lambda_i^4 = \alpha_m \lambda_i^1,\ \lambda_i^5 = \beta_n \lambda_i^2,\ i = 1, \ldots, N.$$

4. ALGORITHMS

Here a certain aspect of the problem related to the numerical solutions to the characteristic equations (25) and (27) corresponding to the free unloaded isotropic plates vibrations is considered.

In reference [6] it has been shown that the solutions set of equation (27) can be divided into two groups. The corresponding vibration modes are related to the symmetric and antisymmetric states of stress.

The antisymmetric vibration modes correspond to the roots of the equations

$$-4\,\mathrm{sh}\frac{r}{2}\mathrm{ch}\frac{s}{2}\,prs + \mathrm{sh}\frac{s}{2}\,\mathrm{ch}\frac{r}{2}\,(r^2+p)^2 = 0, \quad \mathrm{ch}\frac{r}{2} = 0, \qquad (39, 40)$$

where

$$p = \tilde{\alpha}_m^2 + \tilde{\beta}_n^2, \quad r = \sqrt{p-\omega^2}, \qquad (41)$$

and ω is the frequency sought.

Either symmetric or plane vibration modes correspond to the second group of equation (25) solutions. They are represented by the equations

$$-4\,\mathrm{sh}\frac{s}{2}\mathrm{ch}\frac{r}{2}\,prs + \mathrm{sh}\frac{r}{2}\,\mathrm{ch}\frac{s}{2}\,(r^2+p)^2 = 0, \quad \mathrm{sh}\frac{r}{2} = 0. \qquad (42, 43)$$

Therefore, the procedure of the free frequencies vibration determination is simplified and instead of using equations (25) they are found from equation (39)–(43).

Equations (40) and (43) have the explicit solutions

$$\gamma = p + (\pi + 2\pi n)^2, \quad \gamma = p + 4\pi^2 n^2. \qquad (44)$$

It should be mentioned that the procedure of finding the roots of transcendental equations (39) and (42) is extremely difficult without any additional information. In the latter case, one can sometimes find a satisfactory first approximation for a root sought.

Solutions of equations (39) and (42) are sought in the form

$$\gamma = \sum_{n=0}^{\infty} a_n p^n.$$

It can be proved (see reference [7]) that there are approximate relationships for the unknown frequencies (with the accuracy related to small "p"):

1. $\gamma = p[p/[6(1-\nu)] + o(p^2)]$,
2. $\gamma = p$.

3. $\gamma = p[2/(1-\nu) + o(p)]$,
4. $\gamma = p[(2n-1)^2\pi^2/p + 1]$,
5. $\gamma = p[(2n-1)^2\pi^2/p + K_n]$,
   $K_n = 1 + [(1-2\nu)/[2(1-\nu)]]^{1/2}\,16\,\mathrm{ctg}(s/2)/[(2n-1)\pi] + o(p)$,
6. $\gamma = p[4n^2\pi^2/p + 1]$,
7. $\gamma = 2p(1-\nu)/(1-2\nu)[(2n-1)^2\pi^2/p + K_n]$,
   $K_n = 1 + [(1-2\nu)/[2(1-\nu)]]^{1/2}\,16\,\mathrm{ctg}(r/2)/[(2n-1)\pi] + o(p)$,
8. $\gamma = p[4n^2\pi^2/p + K_n]$,
   $K_n = 1 - [(1-2\nu)/[2(1-\nu)]]^{1/2}\,8\,\mathrm{tg}(s/2)/(n\pi) + o(p)$,
9. $\gamma = p[4n^2\pi^2/p + K_n]2(1-\nu)/(1-2\nu)$,
   $K_n = 1 - [(1-2\nu)/[2(1-\nu)]]^{3/2}\,8\,\mathrm{tg}(r/2)/(n\pi)$, (45)

and so on. The above expressions are ordered in the sequence of increasing frequencies.

The importance of the decomposition obtained is mainly expressed during the asymptotic estimation of the frequencies obtained by different applied theories [8].

From a practical point of view, those relationships may be used for small p (for relatively thin plates and low vibration modes) as the approximate expressions for the free vibration isotropic plate frequencies determination. If a higher accuracy is needed, then the values obtained may serve in accuracy improvement by using (for example) Newton's method. However, for large values, Newton's method may be divergent and other numerical methods are recommended.

The relationships (45) analysis shows that equations (39) and (42) have only isolated roots. This means that for each root there exists a neighbourhood which does not include other roots. Therefore, the first step in the determination of the roots is focused on finding intervals consisting of only one root of either equation (39) or (42). The numerical investigation has shown that if one takes $p = 0$ in relationships (45), then the obtained set $\{\bar{\omega}_i(0)\}_{i=1}^{k}$ defines the boundaries of the sought intervals with the isolated roots. To conclude, the problem of finding isolated roots has been solved. The second step includes a direct calculation of the root. The subroutine realizing the calculation algorithm is a part of a developed program for vibration analysis of isotropic plates with the attached masses.

The algorithm of finding the analytic solutions leading to the characteristic equation (25) has been also established. A key part of that investigation lies in analysis of equation (11).

The characteristic equation (25) is given for the case when polynomial (11) roots are simple because of ω^2 and the matrix size, leading to the determination of A, B, C is equal to two.

The free vibration frequency determination of the transversally isotropic plates consists of the following steps.

1. First, the free vibration frequency of plates and shells is calculated with the geometrical parameters, using equations (40) and (43).

2. The interval $(\tilde{G}, \tilde{G})$ is divided into k subintervals where $\{\tilde{G}_i\}_{i=1}^{k}$. $\tilde{G}$ is the relative stiffness parameter $\tilde{G} = G/E$ of an isotropic plate, and $\tilde{G}'$ is the relative stiffness parameter of the transversally isotropic plate.
3. On each step of Newton's method for a fixed $\tilde{G}_i$ value the free vibration transversal isotropic plate frequencies have been found. At the beginning, the frequencies of a free vibration isotropic plate serve as the approximate values and then in each next step the $\tilde{G}$ values are taken from the previous step.
4. After achieving the required $\tilde{G} = \tilde{G}'$ values, the procedure is completed.

It should be emphasized that in spite of simplicity and clarity of the described procedure, its practical realization involves many problems. Among others, they are related to an optimal choice of the optimal $(\tilde{G}', \tilde{G})$ partition, analysis of the obtained results and investigation of convergence rate.

The numerical experiment has shown that the algorithm works stably if the interval [0·01; 0·384] has been divided into more than 20 parts. In order to ensure the required calculation accuracy ($\varepsilon = 10^{-6}$) because of $\tilde{G}$, the number of iterations reaches 10–12.

The described programme of numerical calculations may be successfully used in case of other materials whose physical and geometrical properties lead to equations structurally analogous to equation (25).

5. FREQUENCY SPECTRA: 3-D VERSUS APPROXIMATE THEORIES

The three-dimensional theory is used to solve special problems and to estimate the validity intervals of the two-dimensional theories. This estimation may be made only when the results of two- and three-dimensional theories are compared.

One of the fundamental questions is as follows: which possibilities are used to find approximate solutions to detect free vibration frequencies? The aim of this section is the comparison of the theories for free vibration frequencies detection of an isotropic and transversally isotropic plate with the solution obtained by using the three-dimensional theory. The comparison between two- and three-dimensional theories because of the free vibration frequencies, is carried out on the basis of the following considerations. To each frequency from a spectrum corresponds a vibration mode. The mode analysis possesses an additional positive aspect. The modes may be considered as the characteristics used for the comparison of exact and approximate theories. However, three-dimensional body vibration modes (infinitely many degrees of freedom in the normal direction) and two-dimensional body modes (finite number of degrees of freedom) differ essentially. In reference [7] it is recommended to compare "these plane pictures" obtained by using the two-dimensional theory with the ones obtained by using the three-dimensional theory. In reference [9] it has been shown that for the described technique the parameters of two- and three-dimensional theories are related by

$$2h(\bar{u} + \bar{w},) = \int_{-h}^{h} \bar{u}^* \, \mathrm{d}z; \quad \frac{2h^3}{3}\bar{\gamma} = \bar{n} \int_{-h}^{h} \bar{u}^* z \, \mathrm{d}z, \tag{46}$$

where $\bar{u} = u\bar{i} + v\bar{j}$, $\bar{w} = w\bar{n}$, $\bar{\gamma} = \gamma_x \bar{i} + \gamma_y \bar{j}$.

Parameters related to the three-dimensional theory are marked by a star. It appears that such an averaging of the three-dimensional characteristics is very suitable for a qualitative accuracy asymptotic estimation of the different approximate theories applied to plates analysis. For instance, if the following criterion is used,

$$\lim h^{-h} \left| \frac{R^* - R}{h^m} \right| = O(1)$$

(R, R^* are the compared characteristics obtained by using two- and three-dimensional theories, respectively and m is the two-dimensional theory order of accuracy in comparison with the three-dimensional theory), then m will be equal to 2 for all compared quantities.

Exact analysis realized during the isotropic plate vibrations is much more difficult and practically useless in the case of orthotropic cuboid free vibrations. If one assumes that displacement distribution along the plate's thickness can be predicted on the basis of two-dimensional theory then it is possible to compare normal and tangential distribution obtained for certain characteristic modes.

On the other hand, in reference [10] it is pointed out that if there is a qualitative difference between the vibration modes of the two- and three-dimensional theories used then the two-dimensional theory gives a good approximation of the frequencies spectrum. This information is most practically needed.

Consider now the influence of the geometric parameter λ_1, λ_2 (relative thickness) on a free vibration frequencies spectrum of an isotropic plate.

In Table 2, some calculation results of the vibration frequencies using the classical theory, Timoshenko's type theory (two variants–with and without rotational inertia) for λ_1, λ_2 equal to 5, 10, 50, 100 are given. The frequencies are ordered in an increasing manner (m, n are the integers characterizing the half-wave numbers in the x and y directions, respectively).

The improved theory (with the rotational inertia effects) allows one to pull out three frequency spectra. By using either Kirchhoff's or Timoshenko's theories only one spectrum, related to bending, can be found.

In Table 3, the first 10 frequency spectra by means of three-dimensional theory, are given. In the first row the mode numbers are given, which characterize a wave occurring process in the x and y directions. For the defined m, n the free vibration frequencies are also ordered in an increasing manner.

The dimensionless vibration frequency has been found from the expression valid for both two- and three-dimensional theories:

$$\omega^* = 2h\sqrt{2p(1 + v)/E}\,\omega.$$

When comparing the results given in the tables, focused attention is on the corrections introduced to Kirchhoff's and improved theories (more accurate) for a spectrum related to bending with the different errors for each theory.

TABLE 2

Frequencies ordered in increasing manner obtained for Kirchhoff, Timoshenko and generalized Timoshenko models

	Kirchhoff's theory	Timoshenko's model				Generalized Timoshenko's model		
		Without inertia	With inertia					
m, n	IA	IA	IA	IIA	IIIA	IA	IIA	IIIA
1	2	3	4	5	6	7	8	9
			$G/E = 0{\cdot}384$; $\lambda_1 = 5$, $\lambda_2 = 5$; $v = 0{\cdot}3$					
1 1	0·38527	0·35347	0·34545	3·5762	3·8634	0·34095	2·9380	2·1961
1 2	0·96317	0·79441	0·76487	3·7382	4·3622	0·74789	3·0950	3·6426
2 2	1·5411	1·1643	1·1146	3·8934	4·7895	1·0847	3·2445	4·0184
1 3	1·9264	1·3831	1·3220	3·9935	5·0477	1·2840	3·3404	4·2434
2 3	2·5043	1·6801	1·6048	4·1391	5·4056	1·5556	3·4793	4·5533
1 4	3·2748	2·0317	1·9423	4·3257	5·8407	1·8797	3·6564	4·9276
3 3	3·4674	2·1134	2·0210	4·3711	5·9435	1·9554	3·6993	5·0156
2 4	3·8527	2·2702	2·1726	4·4605	6·1428	2·1012	3·7837	5·1861
3 4	4·8159	2·6312	2·5236	4·6765	6·6106	2·4390	3·9869	5·5848
			$G/E = 0{\cdot}384$; $\lambda_1 = 10$, $\lambda_2 = 10$; $v = 0{\cdot}3$					
1 1	0·096317	0·094131	0·093432	3·4925	3·5712	0·093037	2·8562	2·9282
1 2	0·24079	0·22778	0·22411	3·5346	3·7220	0·22205	2·8974	3·0672
2 2	0·38527	0·35347	0·34545	3·5763	3·8634	0·34095	2·9380	3·1961
1 3	0·48159	0·43335	0·42199	3·6037	3·9533	0·41562	2·9647	3·2777
2 3	0·62606	0·54794	0·53123	3·6446	4·0825	0·52182	3·0044	3·3935
1 4	0·81870	0·69205	0·66799	3·6983	4·2456	0·65429	3·0565	3·5391
3 3	0·86686	0·72669	0·70079	3·7117	4·2840	0·68599	3·0694	3·5741
2 4	0·96317	0·79442	0·76487	3·7382	4·3622	0·74789	3·0950	3·6426
3 4	1·2039	0·95557	0·91722	3·8036	4·5771	0·89478	3·1582	3·8058

TABLE 2 (*continued*)

	Kirchhoff's theory	Timoshenko's model				Generalized Timoshenko's model		
		Without inertia	With inertia					
m, n	IA	IA	IA	IIA	IIIA	IA	IIA	IIIA
1	2	3	4	5	6	7	8	9
			$G/E = 0{\cdot}384$; $\lambda_1 = 50$, $\lambda_2 = 50$; $\nu = 0{\cdot}3$					
1 1	0·0038527	0·0038490	0·0038478	3·4652	3·4685	0·0038471	2·8195	2·8325
1 2	0·0096317	0·0096091	0·0096013	3·4669	3·4751	0·0095970	2·8312	2·8387
2 2	0·015411	0·015353	0·015333	3·4687	3·4816	0·015322	2·8329	2·8448
1 3	0·019263	0·019173	0·015142	3·4698	3·4860	0·019125	2·8340	2·8489
2 3	0·025043	0·024890	0·024839	3·4715	3·4925	0·024809	2·8357	2·8550
1 4	0·032748	0·032489	0·032401	3·4738	3·5011	0·032352	2·8379	2·8631
3 3	0·034674	0·034384	0·034286	3·4743	3·5033	0·034231	2·8385	2·8651
2 4	0·038527	0·038170	0·038049	3·4755	3·5076	0·037981	2·8396	2·8691
3 4	0·048159	0·047602	0·047417	3·4783	3·5183	0·047311	2·8424	2·8791
			$G/E = 0{\cdot}384$; $\lambda_1 = 100$, $\lambda_2 = 100$; $\nu = 0{\cdot}3$					
1 1	0·00096317	0·00096295	0·00096287	3·4644	3·4652	0·00096283	2·8287	2·8295
1 2	0·0024079	0·0024065	0·0024060	3·4648	3·4668	0·0024057	2·8291	2·8310
2 2	0·0038527	0·0038491	0·0038478	3·4652	3·4685	0·0038471	2·8295	2·8325
1 3	0·0048159	0·0048102	0·0048082	3·4655	3·4696	0·0048071	2·8298	2·8336
2 3	0·0062606	0·0062511	0·0062477	3·4660	3·4712	0·0062459	2·8302	2·8351
1 4	0·0081870	0·0081650	0·0081650	3·4665	3·4734	0·0081618	2·8308	2·8371
3 3	0·0086686	0·0086503	0·0086440	3·4667	3·4740	0·0086403	2·8309	2·8377
2 4	0·0096317	0·0096092	0·0096014	3·4669	3·4751	0·0095969	2·8312	2·8387
3 4	0·012040	0·012001	0·011992	3·4677	3·4778	0·011985	2·8319	2·8412

TABLE 3

Frequencies obtained by using three-dimensional theory

	Three-dimensional theory									
m, n	IA	IS	IIS	IIA	IIIA	IIIS	IVS	VS	IVA	VA
1	2	3	4	5	6	7	8	9	10	11
	$G/E = 0{\cdot}384$; $\lambda_1 = 5$, $\lambda_2 = 5$; $v = 0{\cdot}3$									
1 1	0·34207	0·88858	1·4922	3·2648	3·5298	5·6010	6·3457	6·7563	9·4174	9·466
1 2	0·75110	1·4050	2·3320	3·4414	4·0037	5·4795	6·4383	7·1824	9·4184	9·5289
2 2	1·0888	1·7772	2·9070	3·6094	4·0412	5·4635	6·5297	7·5179	9·4315	9·5909
1 3	1·2881	1·9869	3·2123	3·7172	4·6376	5·4894	6·5897	7·7151	9·4460	9·6319
2 3	1·5589	2·2654	3·5870	3·8732	4·9600	5·5727	6·6791	7·9847	9·4753	0·6932
1 4	1·8908	2·5906	3·9654	4·0720	5·3430	5·7549	6·7963	8·3090	9·5269	9·7743
3 3	1·9557	2·6657	4·0423	4·1202	5·4319	5·8112	6·8253	8·3852	9·5419	9·7945
2 4	2·1000	2·8099	4·1789	4·2149	5·6024	5·9346	6·8829	8·5326	9·5741	9·8347
3 4	2·4332	3·1416	4·3718	4·4429	5·9918	6·1870	7·0248	8·8767	9·6672	9·9346
	$G/E = 0{\cdot}384$; $\lambda_1 = 10$, $\lambda_2 = 10$; $v = 0{\cdot}3$									
1 1	0·093150*	0·44429	0·74983	3·1728	3·2465	5·7632	6·2989	6·4461	9·4222	9·4352
1 2	0·22260*	0·70248	1·1827	3·2192	3·3932	5·6652	6·3223	6·6178	9·4193	9·4509
2 2	0·34207*	0·88858	1·4922	3·2648	3·5298	5·6010	6·3457	6·7563	9·4174	9·4666
1 3	0·41714*	0·99346	1·6654	3·2949	3·6160	5·5688	6·3612	6·8384	9·4166	9·4770
2 3	0·52391*	1·1327	1·8936	3·3396	3·7393	5·5315	6·3845	6·9512	9·4162	9·4926
1 4	0·65708	1·2953	2·1569	3·3982	3·8939	5·4969	6·4153	7·0877	9·4170	9·5134
3 3	0·68893*	1·3329	2·2171	3·4126	3·9310	5·4903	6·4230	7·1199	9·4174	9·5185
2 4	0·75110*	1·4050	2·3320	3·4414	4·0037	5·4795	6·4383	7·1824	9·4184	9·5289
3 4	0·89853	1·6019	2·5925	3·5264	4·1766	5·4636	6·4842	7·3294	9·4226	9·5599

TABLE 3 (*continued*)

	Three-dimensional theory									
m, n	IA	IS	IIS	IIA	IIIA	IIIS	IVS	VS	IVA	VA
1	2	3	4	5	6	7	8	9	10	11
	$G/E = 0{\cdot}384$; $\lambda_1 = 50$, $\lambda_2 = 50$; $v = 0{\cdot}3$									
1 1	0·0038474	0·088858	0·15019	3·1428	3·1459	58710	6·2838	6·2915	9·4247	9·4252
1 2	0·0095982	0·14050	0·23745	3·1447	3·1524	5·8618	6·2848	6·3036	9·4245	9·4258
2 2	0·015325	0·17772	0·30032	3·1466	3·1588	5·8532	6·2857	6·3151	9·4243	9·4264
1 3	0·019130	0·19869	0·33575	3·1479	3·1631	5·8477	6·2863	6·3226	9·4242	9·4269
2 3	0·024818	0·22654	0·38278	3·1457	3·1695	5·8398	6·2873	6·3334	9·4241	9·4275
1 4	0·032366	0·25906	0·43767	3·1522	3·1780	5·8297	6·2885	6·3474	9·4239	9·4283
3 3	0·034247	0·26657	0·45034	3·1529	3·1801	5·8273	6·2884	6·3508	9·4238	9·4285
2 4	0·038001	0·28099	0·47467	3·1541	3·1843	5·8226	6·2895	6·3575	9·4237	9·4290
3 4	0·047342	0·32038	0·53062	3·1579	3·1948	5·8112	6·2913	6·3737	9·4234	9·4302
	$G/E = 0{\cdot}384$; $\lambda_1 = 100$, $\lambda_2 = 100$; $v = 0{\cdot}3$									
1 1	0·0009628	0·044429	0·07510	3·1419	3·1427	5·8758	6·2833	6·2859	9·4247	9·4249
1 2	0·0024058	0·070248	0·11874	3·1424	3·1443	5·8733	6·2836	6·2884	9·4247	9·4250
2 2	0·0038473	0·088858	0·15019	3·1428	3·1459	5·8710	6·2838	6·2915	9·2447	9·4252
1 3	0·0048075	0·099345	0·16791	3·1432	3·1470	5·8694	6·2840	6·2936	9·4246	9·4253
2 3	0·0062464	0·11327	0·19145	3·1436	3·1486	5·8671	6·2842	6·2966	9·4246	9·4255
1 4	0·0081628	0·12953	0·21892	3·1443	3·1508	5·8641	6·2845	6·3006	9·4245	9·4257
3 3	0·0086414	0·13329	0·22526	3·1444	3·1513	5·8633	6·2846	6·3016	9·4245	9·4257
2 4	0·0095982	0·14050	0·23745	3·1447	3·1524	5·8618	6·2848	6·3036	9·4245	9·4258
3 4	0·011987	0·15708	0·26546	3·1455	3·1551	5·8582	6·2852	6·3084	9·4244	9·4261

*According to the values obtained in reference [10].

It is remarkable that the approximation nearness of the frequency values obtained for all theories are applied for $\lambda_1 = 50$, 100. In the case of λ_1, the error is smaller than 2%, whereas in the case of the Timoshenko model it is 0·5%. The general Timoshenko model gives most accurate results. In the latter case the results were the same (the difference can be noticed in the fourth decimal place). The results obtained show that there exists no essential influence of a transversal displacement on the frequencies of the modes considered related to bending.

With an increase in a plate's thickness(λ_1 decreases) Kirchhoff's theory is less accurate when finding the frequencies related to bending. For $\lambda_1 \geqslant 10$ the fundamental bending vibration frequency, found by using the classical model, does not reach the threshold of 5%, whereas for λ_1 it is equal to 5–12%.

During calculations of higher frequencies the application area of the classical theory is defined on the basis of comparison of the corresponding modes with the three-dimensional theory results. With an increase of m, n, an increase in the error is obtained: from 12% ($m = n = 2$) to 34% ($m = 3$, $n = 4$). For $\lambda_1 = 5$ an error of 10% is admitted only during the fundamental frequency estimation, whereas for $m = 3$, $n = 4$ (ninth mode) an error is equal to 100%.

The accuracy increase of Timoshenko's type theory allows one to get practically exact values of the isotropic plate vibration frequencies.

In the considered intervals of the λ_1, λ_2 parameters changes of an error introduced by Timoshenko's theory reached 0·5%. As lightly larger error characterizes vibration frequencies related to bending (without rotary inertial effects): for $\lambda_1 = 5$ one has 3% ($m = n = 1$) and 8% ($m = 3$, $n = 4$). If the governing equations introduce the rotary inertial effects then the frequencies are obtained with the error 4%.

It should be emphasized that the classical and the improved Timoshenko's theories give frequencies with higher values than the exact results. The general Timoshenko model gives lower values than the exact results. Therefore, with increasing plate thickness, the difference between frequencies obtained by using classical theory and the improved Timoshenko theory is also increased. In the case of the general Timoshenko theory, the error decreases with thickness and then slightly increases. However, the error oscillations are of about 0·3%.

Next an analysis of the successive frequency spectra obtained by the improved two-dimensional theories is presented. The frequencies related to thickness-rotational modes (IIA) and thickness-displacements modes (IIIA) of a freely vibrating isotropic plate belong to them. Here more evident error divergence is observed, in comparision with the exact three-dimensional theory results as well as in relation to the "improved" two-dimensional theories described in this work. The calculation error does not depend practically on the magnitude of the relative thickness parameter λ_1 and oscillates in the interval from 4 to 10% for both vibration modes. In reference [11] during the formulation of the two-dimensional improved theory a displacement coefficient was equal to $\pi^2/12$. This value secures overlap of the low frequency spectra. The corresponding modes are thickness-rotary type with $w = 0$. Therefore, the displacement coefficient $\pi^2/12$ gives the best approximation for the stress deformation state, where the transversal displacements play a key role. There exists also another criterion for choosing the displacement coefficient [2].

When using "improved" theories in this work the displacement coefficient was equal to 1 (Timoshenko's model) and 2/3 (general Timoshenko's model). Therefore, following inequalities hold for the "improved" frequencies "ω_T" and "ω_{0T}" and the exact one ω:

$$\omega_{0T} < \omega < \omega_T. \tag{47}$$

This means that the improved theories allow one to find intervals where "real" vibration frequencies occur. A comparison between exact and approximate frequency values, corresponding to thickness-rotary and thickness-displacement modes, found by two- and three-dimensional theories indicates validity of the relationship

$$\omega' = (\omega^T + \omega_{0T})/2, \tag{48}$$

where a frequency value close to the exact one is obtained.

The frequencies obtained from equation (48) differ from the exact values by less than 2·5%.

As has been mentioned earlier, the essential difference between two- and three-dimensional theories consists of qualitative difference in their spectra. The three-dimensional theory allows for an infinite set of spectra definition, whereas the two-dimensional theory possesses a finite number of spectra, which are equal to the degrees of freedom of a two-dimensional model of a continuous medium. The free vibration frequencies corresponding to the modes with the numbers IIIS, IVS, VS, VA are not "caught" by the two-dimensional theory. In order to find them, applicable theories, characterized by less requirements with regard to the kinematic hypotheses and having more degrees of freedom (they are sometimes called higher order theories), should be formulated.

Consider the fundamental results of comparing the vibration frequency distributions obtained by using three-dimensional theories (see Table 3). For $\lambda_1 \leqslant 10$ and for higher vibration modes the frequency distribution is more uniform. With the decreasing of a plate's thickness (increasing λ_1 and λ_2) the distribution picture becomes principally changed. For $p \to 0$ ($p = (\alpha_m/\lambda_1)^2 + (\beta_n/\lambda_2)^2$) frequencies approaching, for instance IIA and IIIA, IVS and VS, and so on are observed. The frequencies related to bending and frequency spectra of the symmetric modes tend to approach the zero limit, and a convergence is characterized by the first terms of the asymptotic series. The convergence effect can be foreseen by using the modified Timoshenko's theories for frequency spectra IIA and IIIA.

The material properties influence on the free vibration frequencies will be analyzed by using an isotropic-transversal body, whose isotropic plane coincides with $x0y$ plane. This limitation of the considerations leads to the reduction of the parameters' number and only the specific parameters are left.

Consider the influence of G'/E on the vibration frequency spectrum of a transversally isotropic plate for different values of λ_1, λ_2 characterizing the relative thickness.

In Table 4 the frequencies obtained by using two-dimensional classical theory, Timoshenko type theory (with and without inertia rotary effects) are given (in the

TABLE 4

Frequencies obtained using Kirchhoff, Timoshenko and generalized Timoshenko models

m, n	Kirchhoff's *theory*	Timoshenko's model				Generalized Timoshenko's model		
		Without inertia	*With inertia*					
1	2	3	4	5	6	7	8	9
$G'/E = 0{\cdot}01;\ \lambda_1 = 5,\ \lambda_2 = 5;\ v = 0{\cdot}3$								
1 1	0·38527	0·13429	0·13423	1·0496	1·6033	0·12390	0·91633	1·4181
1 2	0·96317	0·22053	0·22047	1·5119	2·4402	0·20243	1·3368	2·1710
2 2	1·5412	0·28173	0·28169	1·8629	3·0559	0·25811	1·6537	2·7231
1 3	1·9264	0·31604	0·31600	2·0639	3·4050	0·28939	1·8347	3·0358
2 3	2·5042	0·36146	0·36143	2·3333	3·8702	0·33083	2·0769	3·4523
1 4	3·2748	0·41437	0·41434	2·6502	4·4148	0·37910	2·3616	3·9396
3 3	3·4674	0·42657	0·42654	2·7236	4·5407	0·39024	2·4275	4·0523
2 4	3·8527	0·44999	0·44996	2·8649	4·7827	0·41161	2·5543	4·2688
3 4	4·8159	0·50379	0·50376	3·1909	5·3398	0·46073	2·8467	4·7672
$G'/E = 0{\cdot}01;\ \lambda_1 = 10,\ \lambda_2 = 10;\ v = 0{\cdot}3$								
1 1	0·096317	0·057482	0·057422	0·71372	0·93692	0·054084	0·60491	0·81221
1 2	0·24079	0·10249	0·10243	0·89749	1·3131	0·095029	0·77639	1·1566
2 2	0·38527	0·13429	0·13423	1·0496	1·6032	0·12390	0·91633	1·4181
2 3	0·62606	0·17534	0·17528	1·2629	1·9951	0·16121	1·1110	1·7711
1 4	0·81870	0·20238	0·20233	1·4106	2·2602	0·18582	1·2451	2·0094
3 3	0·86686	0·20860	0·20855	1·4452	2·3217	0·19148	1·2764	2·0646
2 4	0·96317	0·22053	0·22048	1·3119	2·4402	0·20234	1·3368	2·1710
3 4	1·2039	0·24786	0·24781	1·6672	2·7138	0·22724	1·4771	2·4164

TABLE 4 (*continued*)

m, n	Kirchhoff's *theory*	Timoshenko's model				Generalized Timoshenko's model		
		Without inertia			*With inertia*			
1	2	3	4	5	6	7	8	9
$G'/E = 0{\cdot}01$; $\lambda_1 = 50$, $\lambda_2 = 50$; $v = 0{\cdot}3$								
1 1	0·0038527	0·0037205	0·0037194	0·56559	0·57858	0·0036947	0·46294	0·47557
1 2	0·0096317	0·0088639	0·0088586	0·57597	0·60732	0·0087266	0·47307	0·50337
2 2	0·015411	0·013573	0·013562	0·58616	0·63472	0·013269	0·48298	0·52968
1 3	0·019263	0·016509	0·016494	0·59286	0·65234	0·016076	0·48947	0·54650
2 3	0·025043	0·020655	0·020634	0·60276	0·67790	0·020009	0·49906	0·57078
1 4	0·032749	0·025772	0·025745	0·61573	0·71052	0·024825	0·51156	0·60162
3 3	0·034674	0·026988	0·026959	0·61892	0·718444	0·025963	0·51463	0·60908
2 4	0·038527	0·029351	0·029328	0·62526	0·73402	0·028171	0·52074	0·62374
3 4	0·048159	0·034903	0·034863	0·64075	0·77158	0·033332	0·53568	0·65893
$G'/E = 0{\cdot}01$; $\lambda_1 = 100$, $\lambda_2 = 100$; $v = 0{\cdot}3$								
1 1	0·00096317	0·0009546	0·00095451	0·56033	0·56364	0·00095282	0·45780	0·46103
1 2	0·0024079	0·0023533	0·0023548	0·56297	0·57116	0·0023448	0·46038	0·46836
2 2	0·0038527	0·0037205	0·0037194	0·56559	0·57858	0·0036947	0·46294	0·47557
1 3	0·0048159	0·0046120	0·0046104	0·56733	0·58347	0·0045727	0·46465	0·48032
2 3	0·0062606	0·0059224	0·0059199	0·56994	0·59073	0·0058587	0·46719	0·48735
1 4	0·0081870	0·0076223	0·0076183	0·57339	0·60026	0·0075192	0·47056	0·49657
3 3	0·0086686	0·0080393	0·0080348	0·57425	0·60262	0·0079252	0·47139	0·49885
2 4	0·0096317	0·0088630	0·0088546	0·57597	0·60732	0·0087267	0·47307	0·50337
3 4	0·012040	0·010874	0·010866	0·58024	0·61889	0·010673	0·47722	0·51450

first row the mode numbers are presented). The parameters used are $\lambda_1 = \lambda_2 = 5$, 10, 50, 100. For the parameters and the modes given in Table 5 the frequency values obtained by using the three-dimensional theory have been given (the first 10 spectra arranged in the increasing manner for each pair of the wave numbers).

An essential influence on the frequency spectrum of the transversally isotropic free vibrations (thick or thin) involves a transversal displacement. It increases with the increase of mode number and relative plate thickness. Rotary inertia effects slightly decrease the frequencies (to 0·1%) for different vibrations and different modes of thin ($\lambda_1 = 50$, 100), average ($\lambda_1 = 10$) and thick ($\lambda_1 = 5$) plates. This leads to the conclusion that the rotary inertia influence is smaller than that in the case of an isotropic plate material.

Consider now the drawbacks and the advantages of the classical and improved Timoshenko's theories.

In the case considered, the classical theory possesses a narrower application area than for the calculation of an isotropic plate frequency. For $\lambda_1 = 100$ an error less than 5% can be obtained only during the calculations of the first three frequencies corresponding to the numbers $m = n = 1$, $m = 1$, $n = 2$; $m = n = 2$. For $s > 8$ (where "s" is the ordinal frequencies number in the bending spectrum) the error is greater than 10%. For $\lambda_1 = 50$ the error is smaller than 5% when using the classical methods, whereas for $\lambda_1 = 5$, 10 the Kirchhoff theory leads to incorrect results. For instance, for $\lambda_1 = 5$, the fundamental frequency determination error is equal to 200%, and during the ninth mode determination it exceeds 900%!

The results obtained by using the Timoshenko kinematic model differ slightly from the exact results obtained by using three-dimensional theory. The error does not exceed 6% for a fundamental frequency ($\lambda_1 = 5$), and even decreases to 3% with an increase of mode number. Also, the results obtained by using the general Timoshenko model are close to exact. For $\lambda_1 = 100$ the latter theory gives practically the exact free vibration frequency values. With the increase of thickness and the mode numbers error increases, but for the changes considered in parameter intervals error does not exceed 6%.

Another important and interesting problem is related to the possibility of the application of modified theories to the free vibration frequencies determination of the successive spectra. It is obvious that between frequencies defined by using the modified theories and the three-dimensional theories is the same relationship as for a material isotropy. If one takes equation (48), then one gets frequency values close to the real values, but the interval (ω_{0T}, ω_T) includes the frequencies corresponding to different vibration modes.

The frequency calculation results analysis when using the three-dimensional theory show that the frequency spectra of a free vibration transversally isotropic plate are more continuous and more uniform than those of an isotropic plate spectra. For instance, the frequency values of the first 10 spectra of a three-dimensional plate lie in the interval (0, 5) for a transversally isotropic material; for an isotropic plate that interval corresponds to (0, 10) and the frequencies are shifted to the left.

In the transversally isotropic plate frequency interval there are also frequencies of the isotropic plate vibrations. It seems to be strange at first glance, however,

TABLE 5

Frequencies obtained by using three-dimensional theory

Three-dimensional theory										
1	2	3	4	5	6	7	8	9	10	11
$G'/E = 0{\cdot}01$; $\lambda_1 = 5$, $\lambda_2 = 5$; $v = 0{\cdot}3$										
1 1	0·12695	0·14328	0·88858	1·0288	1·3476	1·4922	1·5823	1·7604	1·8031	2·1314
1 2	0·21058	0·22654	0·4050	1·4935	1·7322	1·0696	2·3309	2·4179	2·4657	2·5615
2 2	0·27072	0·28656	1·7772	1·8479	2·0457	2·3383	2·6952	2·9023	3·0257	3·0941
1 3	0·30459	0·32038	1·9869	2·0505	2·2303	2·4766	2·7881	3·1468	3·2123	3·3958
2 3	0·34955	0·36529	2·2654	2·3214	2·4817	2·7279	3·0394	3·3982	3·5755	3·7908
1 4	0·40203	0·41773	2·5906	2·6397	2·7817	3·0035	3·2889	3·6231	3·9550	3·9936
3 3	0·41415	0·42984	2·6657	2·7134	2·8518	3·0685	3·3484	3·6771	4·0332	4·0428
2 4	0·43742	0·45809	2·8099	2·8552	2·9870	3·1946	3·4643	3·7830	4·1392	4·1732
3 4	0·49092	0·50657	3·1416	3·1822	3·3009	3·4899	3·7384	4·0355	4·3712	4·4429
$G'/E = 0{\cdot}01$; $\lambda_1 = 10$, $\lambda_2 = 10$; $v = 0{\cdot}3$										
1 1	0·054467	0·071639	0·44429	0·67380	0·74983	0·90580	1·1063	1·2587	1·5833	1·6937
1 2	0·056670	0·11327	0·70248	0·86608	1·1827	1·2329	1·2896	1·5553	1·6742	1·9252
2 2	0·12695	0·14328	0·88858	1·0228	1·3476	1·4922	1·5823	1·7604	1·8031	2·1314
1 3	0·14397	0·16019	0·99346	1·1152	1·4189	1·6653	1·7501	1·8156	1·9506	2·6256
2 3	0·16653	0·18265	1·1327	1·2408	1·5197	1·8936	1·9901	2·0636	2·2113	2·4134
1 4	0·19285	0·20886	1·2953	1·3908	1·6445	1·9968	2·1562	2·2391	2·3949	2·4049
3 3	0·19892	0·21492	1·3329	1·4259	1·6742	2·0214	2·2163	2·3003	2·4253	2·4517
2 4	0·21058	0·22654	1·4059	1·4935	1·7322	2·0696	2·3309	2·4179	2·4657	2·5615
3 4	0·23738	0·25328	1·5708	1·6505	1·8992	2·1150	2·5101	2·6201	2·6607	2·7321

				$G'/E = 0{\cdot}01;\ \lambda_1 = 50,\ \lambda_2 = 50;\ v = 0{\cdot}3$						
1 1	0·0036950	0·014329	0·88858	0·15019	0·51430	0·52847	1·0170	1·0241	2·5344	2·5372
1 2	0·0087293	0·022654	0·14050	0·23745	0·52569	0·55968	1·0228	1·0403	2·5367	2·5438
2 2	0·013277	0·028656	0·17772	0·30032	0·53684	0·58921	1·0286	1·0563	1·5301	1·5489
1 3	0·016089	0·032038	0·19869	0·33575	0·54414	0·60808	1·0324	1·0668	1·5326	1·5561
2 3	0·020034	0·036529	0·22654	0·38278	0·55492	0·63533	1·0382	1·0824	1·5365	1·5668
1 4	0·024870	0·041773	0·25906	0·43767	0·56897	0·66991	1·0457	1·1028	1·5416	1·5810
3 3	0·026014	0·042984	0·26657	0·45034	0·57243	0·67828	1·0476	1·1079	1·5429	1·5846
2 4	0·018234	0·045309	0·28099	0·47468	0·57928	0·69471	1·0514	1·1179	1·5455	1·5916
3 4	0·033433	0·050657	0·31416	0·53062	0·59608	0·73415	1·0607	1·1426	1·5518	1·6091
				$G'/E = 0{\cdot}01;\ \lambda_1 = 100,\ \lambda_2 = 100;\ v = 0{\cdot}3$						
1 1	0·00095288	0·0071639	0·044429	0·75097	0·50851	0·51213	1·0141	1·0159	2·5332	2·5339
1 2	0·0023449	0·011327	0·070248	0·11874	0·51141	0·52037	1·0156	1·0200	2·5334	2·5356
2 2	0·0036950	0·014328	0·088858	0·15019	0·51143	0·52847	1·0170	1·0241	2·5344	2·5372
1 3	0·0045732	0·016019	0·099346	0·16791	0·51622	0·53380	1·0180	1·0268	2·5348	2·5383
2 3	0·0058597	0·018264	0·11327	0·19144	0·51908	0·54170	1·0195	1·0309	2·5354	2·5400
1 4	0·0075210	0·020886	0·12953	0·21892	0·52286	0·55205	1·0214	1·0363	2·5361	2·5422
3 3	0·0079273	0·021492	0·13329	0·22526	0·52381	0·55460	1·0210	1·0376	2·5363	2·5427
2 4	0·0087293	0·022654	0·14050	0·23745	0·52569	0·55968	1·0228	1·0403	2·5367	2·5438
3 4	0·010677	0·025328	0·15708	0·26546	0·53036	0·57217	1·0252	1·0470	2·5377	2·5466

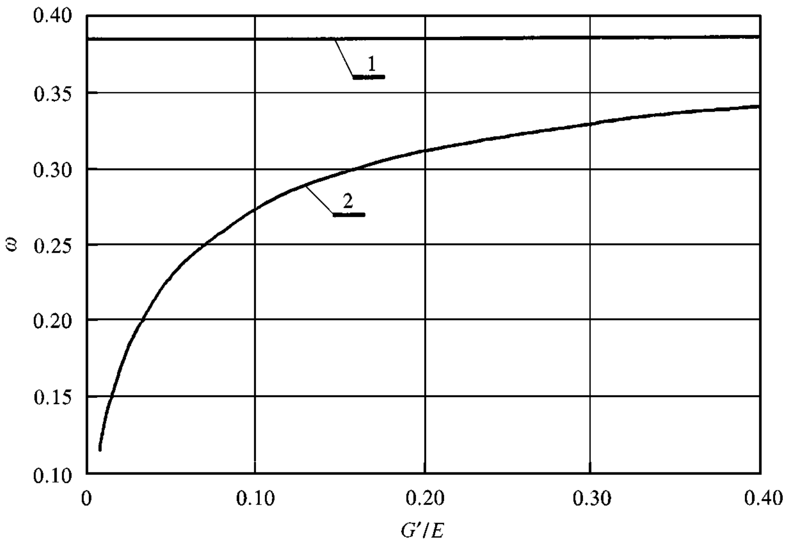

Figure 1. 3-D theory versus 2-D approximate theory for free orthotropic (isotropic) plate vibrations; Dependence on G'/E.

because according to the physical and geometrical choice of the parameters of the transversal-isotropic material, low transversal stiffness does not influence the frequencies related to the tension-compression vibrations.

In Figure 1 the frequency against G'/E relation for $\lambda_1 = \lambda_2 = 5$ is presented. Curve 1 corresponds to the fundamental frequency found by using the three-dimensional theory. The curves obtained by using the improved Timoshenko theory are not practically distinguishable from curve 2.

The presented curves clearly characterize the G'/E influence on the free vibration fundamental frequency. For G'/E close to 0·384 values (isotropic material), the results obtained indicate a slight influence of transversal stiffness on the fundamental frequency. With the decrease of G'/E the fundamental frequency is reduced.

6. NUMERICAL INVESTIGATION OF THE ADDED MASSES INFLUENCE

In the previous sections, a comparison between the results obtained by using modified and three-dimensional theories of the free isotropic and transversally isotropic plates vibrations has been conducted. A similar approach is highly required for plates with attached masses.

While searching for the solution to the problems related to plates and shells, there are principally two possibilities: exact problem formulation in the frame of the three-dimensional elasticity theory and definition of the possible steps of solutions or definition of the approximated calculation models for the different problems. The second approach is of more interest in technical applications.

The fundamental results relating to the free plates and shells vibrations with the attached masses have been achieved by using the second approach realized by the following model: a plate (Kirchhoff–Love model) being absolutely hard mass (concentrated or distributed) and vertically vibrating. A tendency towards the application of this model is observed in the case of forced vibrations [12–14], parametric vibrations [15, 16], stochastic vibrations [17], non-linear vibrations [18, 19] as well as during the dynamic stability analysis [20, 21] and optimization [22, 23] of plates and shells with the added masses. On the other hand, the first approach gives a better possibility of the physical interpretation of a problem and seems to be a strong tool for building new models as well as calculation algorithms. It allows for the analysis of existing models and development of approximate methods applied directly by engineers. It also allows for the development of investigated objects. However, it requires a high development of computer techniques.

If one takes into account that the numerical investigation of the dynamical characteristics with the attached masses is not a simple task, then one can imagine how difficult it is to solve similar problems in the frame of three-dimensional theories.

Problems related to the analytical results, the lack of optimal numerical investigations and high consuming time of numerical as well as symbolic computations do not allow for receiving the general formulas. It does not give any optimistic prognosis for the dynamic characteristic definition of the analyzed systems.

Therefore, a compromise solution should change the complex analytic formulas using the numerical calculations.

A traditional analysis causes many calculation difficulties in finding solutions to the frequency equations of the "shell–mass" systems. As has been shown in reference [24], the free vibration frequencies of elastic systems with discrete added elements may be found as the eigenvalues of a certain class of matrix with trigonometric series as its elements. In our case we obtain the transcendental equation

$$4\omega^2 M \sum_{m,n}^{\infty} w_z^0 \sin^2 \alpha_m x_1 \sin^2 \beta_m y_1 + 1 = 0, \tag{49}$$

where

$$w_z^0 = \frac{\Delta_1}{\Delta} d_{13}^{(1)} \operatorname{sh} 0{\cdot}5\tau_1 + \frac{\Delta_2}{\Delta} d_{13}^{(1)} \operatorname{ch} 0{\cdot}5\tau_1 + \frac{\Delta_3}{\Delta} d_{13}^{(2)} \operatorname{ch} 0{\cdot}5\tau_2$$
$$+ \frac{\Delta_4}{\Delta} d_{13}^{(2)} \operatorname{sh} 0{\cdot}5\tau_2 + \frac{\Delta_5}{\Delta} d_{13}^{(3)} \operatorname{ch} 0{\cdot}5\tau_3 + \frac{\Delta_6}{\Delta} d_{13}^{(3)} \operatorname{sh} 0{\cdot}5\tau_3, \tag{50}$$

Δ is the determinant of the linear equations (24), and Δ_i is the minors obtained from Δ by cancellation of the lth row and ith column.

In the case of the isotropic plate one finds the following explicit formula for w_z^0:

$$w_z^0 = [\omega^2 s(\operatorname{sh} r \operatorname{ch} s(r^2 + p)^2 - 4\,\mathrm{pr}\, s \operatorname{ch} r \operatorname{ch} s)]/\Delta. \tag{51}$$

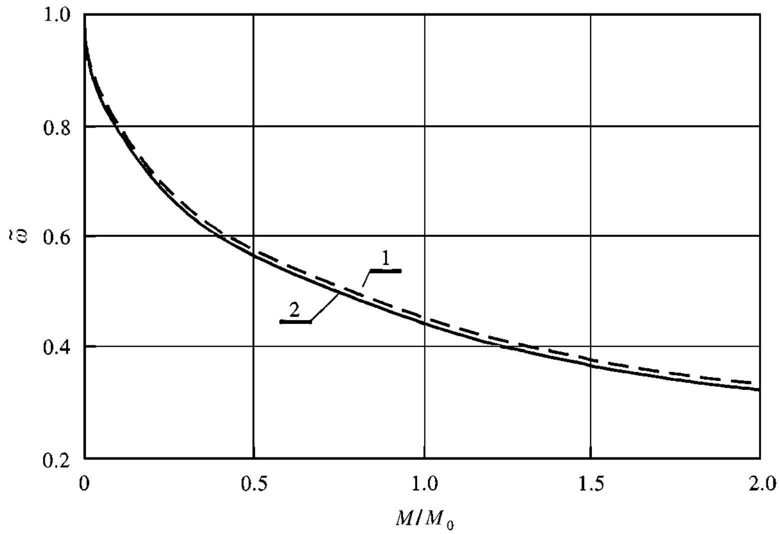

Figure 2. As Figure 1 but dependence on M/M_0.

Here

$$\Delta = 8(r^2 + p)^2 \,\mathrm{pr}\, s(1 - \mathrm{ch}\, r\, \mathrm{ch}\, s) + \mathrm{sh}\, r\, \mathrm{sh}\, s[16p^2r^2s^2 + (r^2 + p)^4],$$

$$p = \tilde{\alpha}_m + \tilde{\beta}_n^2, \quad \tilde{\alpha}_m = \alpha_m/\lambda_1, \quad \tilde{\beta}_n = \beta_n/\lambda_2,$$

$$r = \sqrt{p - \omega^2}, \quad s = \sqrt{p - (1 - 2\nu)\omega^2/(2 - 2\nu)}.$$

and ω is the frequency sought.

The received assumptions allow one to retain only the component of the essential influence. The obtained equations are more simplified.

The equations obtained finally are solved by using the algorithm described in reference [25]. The only difference is that in the subroutine realizing the numerical calculations of the roots of equation (49), there is no second step included and the frequencies of the orthotropic plate free vibrations are taken as the initial values. The suitability of writing two independent programs is substantiated by lower time for the free vibration frequencies computation for an isotropic plate with the discrete added masses.

Consider the influence of the geometric and physical parameters as well as the added masses value on the frequency spectra of the free vibrations of an isotropic plate and compare the results obtained by using the classical and the three-dimensional theories.

In Figure 2 a frequency $\tilde{\omega}$ dependence

$$\tilde{\omega} = \omega_M/\omega_0, \tag{52}$$

TABLE 6

Frequencies of the isotropic plate with attached mass: TK – Kirchhoff's theory; MT–Timoshenko's model (a) *including inertia of rotation* (b) *without inertia of rotation*; GT – *generalized Timoshenko's model*; TT – *three dimensional theory*

		MT			
M/M_0	TK	a	b	GT	TT
		$G/E = 0{\cdot}384$; $\lambda_1 = 5$, $\lambda_2 = 5$; $v = 0{\cdot}3$			
0·01	0·37785	0·34666	0·33908	0·33464	0·33602
0·1	0·32538	0·29955	0·29316	0·28906	0·29164
0·5	0·22052	0·20314	0·19897	0·19575	0·19822
1·0	0·17005	0·15709	0·15331	0·15069	0·15262
2·0	0·12631	0·11627	0·11378	0·11177	0·11318
		$G/E = 0{\cdot}384$; $\lambda_1 = 10$, $\lambda_2 = 10$; $v = 0{\cdot}3$			
0·01	0·094463	0·092318	0·091658	0·091270	0·091404
0·1	0·081344	0·079457	0·079037	0·078690	0·078942
0·5	0·055130	0·053733	0·053605	0·053443	0·053664
1·0	0·042512	0·041389	0·041332	0·041120	0·041407
2·0	0·031577	0·030719	0·030696	0·030534	0·030766
		$G/E = 0{\cdot}384$; $\lambda_1 = 50$, $\lambda_2 = 50$; $v = 0{\cdot}3$			
0·01	0·0037785	0·0037750	0·0037739	0·0037731	0·0037735
0·1	0·0032538	0·0032502	0·0032499	0·0032493	0·0032501
0·5	0·0022052	0·0022013	0·0022026	0·0022022	0·0022034
1·0	0·0017005	0·0016969	0·0016985	0·0016981	0·0017004
2·0	0·0012631	0·0012602	0·0012616	0·0012613	0·0012661
		$G/E = 0{\cdot}384$; $\lambda_1 = 100$, $\lambda_2 = 100$; $v = 0{\cdot}3$			
0·01	0·00094463	0·00094441	0·00094434	0·00094429	0·00094431
0·1	0·00081344	0·00081323	0·00081320	0·00081316	0·00081320
0·5	0·00055130	0·00055107	0·00055114	0·00055111	0·00055113
1·0	0·00042512	0·00042491	0·00042499	0·00042497	0·00042501
2·0	0·00031577	0·00031560	0·00031568	0·00031566	0·00031572

against M/M_0 for a lower isotropic plate vibration frequency is given. Here ω_0 is the free vibration frequency of the isotropic plate (without any additional mass) ω_M is the frequency (corresponding to a plate with an added mass) with the values given in Table 6 for the different geometric parameters λ_1, λ_2. The additional mass position has the following co-ordinates: $x_1 = 0{\cdot}476$; $y_1 = 0{\cdot}476$.

The results, given in Figure 2, define a change of the parameter $\tilde{\omega}$ corresponding to the fundamental plate vibration frequency versus the added mass M for relative thickness parameter $\lambda_1 = 5$. The results obtained by using Kirchhoff's theory (curve 1), the improved Timoshenko theory (curve 2) defined by $\tilde{\omega} = \tilde{\omega}(M/M_0)$ are practically indistinguishable in Figure 2. From the data of Figure 2, it can be concluded that both approximate and three-dimensional theories give similar results. This conclusions is also true for $\lambda_1 \geqslant 5$.

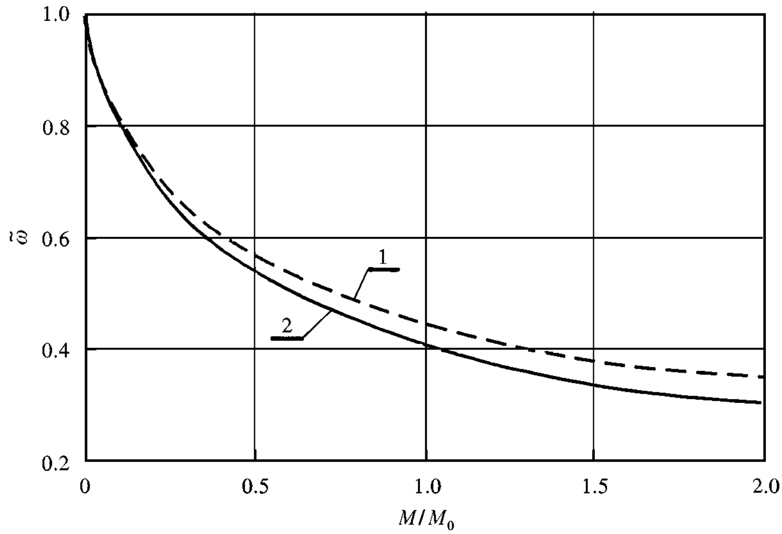

Figure 3. As Figure 2, but see text for different meanings of the curves *1*, *2* in the two figures.

Upon carrying out the analysis of the parameter related to frequency (Figure 2) against the added mass the following conclusions are derived.

1. $\tilde{\omega}$ decreases with increase of the relative thickness parameter. Therefore, the fundamental free vibration frequency of a plate with and without the attached masses for a given λ_1 decreases.

2. In the case of the attached masses the frequencies $\tilde{\omega}$ decrease with the increase of M/M_0 for a given λ_1. With the change of M/M_0 in the interval 0·01–2·0 the frequencies $\tilde{\omega}$ decrease correspondingly: 1·02 times for $M/M_0 = 0{\cdot}01$; 1·18 times for $M/M_0 = 0{\cdot}1$; 1·74 times for $M/M_0 = 0{\cdot}5$; 2·25 times for $M/M_0 = 1{\cdot}0$; 3·03 times for $M/M_0 = 2{\cdot}0$. From the given data one can conclude that during the change of the physical and geometrical plate parameters the attached mass influence on free vibration fundamental frequency of an isotropic plate is essential.

One can now focus on a composite plate made from a transversal isotropic material with each point having an isotropic plane parallel to the $x0y$ plane. In order to decrease the number of control parameters only the specific properties characterizing the material with a low transversal stiffness are left.

For a plate made of a transversally isotropic material the following physical and geometrical parameters are taken: $G'/E = 0{\cdot}01$; $\lambda_1 = \lambda_2 = 5$; 10; 50; 100. The attached mass has the co-ordinates: $x_1 = y_1 = 0{\cdot}476$.

In Figure 3 the change of the control parameter $\tilde{\omega} = \omega_M/\omega_0$, where ω_0 is the free vibration frequency of a transversally isotropic plate (without an added mass), ω_M is a frequency of a loaded plate (they are given in Table 7 for the different geometric parameters λ_1, λ_2) for different values of a ratio M/M_0 for the fundamental vibration frequency of a transversal isotropic plate with a concentrated mass. The following notations are used: 1—the $\tilde{\omega}$ the curve obtained by using a classical theory; 2—the curve built by using Timoshenko type theory, the generalized

TABLE 7

Frequencies of transversally isotropic plate with a concentrated mass: TK – Kirchhoff's theory; MT–Timoshenko's model including (a) *inertia of rotation* (b) *without inertia of rotation*; GT – *generalized Timoshenko's model*; TT – *three dimensional theory*

		MT			
M/M_0	TK	a	b	GT	TT
		$G'/E = 0{\cdot}01$; $\lambda_1 = 5$, $\lambda_2 = 5$; $\nu = 0{\cdot}3$			
0·01	0·37785	0·13168	0·13159	0·12147	0·12446
0·1	0·32538	0·11212	0·11209	0·10341	0·10615
0·5	0·22052	0·072811	0·072802	0·067081	0·068994
1·0	0·17005	0·055128	0·055124	0·050771	0·052242
2·0	0·12631	0·040445	0·040443	0·037240	0·038444
		$G'/E = 0{\cdot}01$; $\lambda_1 = 10$, $\lambda_2 = 10$; $\nu = 0{\cdot}3$			
0·01	0·094463	0·056368	0·056311	0·053036	0·053399
0·1	0·081344	0·048155	0·048120	0·045288	0·045770
0·5	0·055130	0·031628	0·031619	0·029691	0·030092
1·0	0·042512	0·024049	0·024045	0·022560	0·022789
2·0	0·031577	0·017692	0·017690	0·016588	0·016810
		$G'/E = 0{\cdot}01$; $\lambda_1 = 50$, $\lambda_2 = 50$; $\nu = 0{\cdot}3$			
0·01	0·0037785	0·0036489	0·0036479	0·0036235	0·0036225
0·1	0·0032538	0·0031397	0·0031391	0·0031177	0·0031183
0·5	0·0022052	0·0021209	0·0021207	0·0021050	0·0020995
1·0	0·0017005	0·0016328	0·0016327	0·0016201	0·0016184
2·0	0·0012631	0·0012114	0·0012114	0·0012018	0·0012023
		$G'/E = 0{\cdot}01$; $\lambda_1 = 100$, $\lambda_2 = 100$; $\nu = 0{\cdot}3$			
0·01	0·00094463	0·00093621	0·00093613	0·00093414	0·00093415
0·1	0·00081344	0·00080603	0·00080598	0·00080452	0·00080459
0·5	0·00055130	0·00054580	0·00054578	0·00054470	0·00054478
1·0	0·00042512	0·00042069	0·00042069	0·00041982	0·00041988
2·0	0·00031577	0·00031239	0·00031239	0·00031172	0·00031176

Timoshenko model and the three-dimensional theory (practically in the figure's scale they are not distinguished). The calculation results given in Figure 3 allows one to draw conclusions about the frequency spectra of the transversally isotropic plate with the attached mass as well as some conclusions about the relative dependencies obtained using three-dimensional and approximate theories for certain values of the physical and geometrical parameters and the attached mass values.

The free vibration frequencies of the transversally isotropic plate with and without attached mass are lower than for a similar but isotropic plate. Such results are given by the three-dimensional theory and the improved theories.

The occurrence of the additional mass decreases a fundamental frequency value of the transversally isotropic plate. If as a control parameter one takes ω_M^*/ω_0^*; then for the plates with small thickness ($\lambda_1 = 50$, 100) the dependence $\tilde{\omega} = \tilde{\omega}(M/M_0)$

(transversally isotropic material) will be practically indistinguishable from a similar dependence for the isotropic plate with the attached mass (for both two and three-dimensional theories applied). With an increase of the attached mass value, an error introduced by classical and improved theories (for $\lambda_1 < 50$) slightly increases in comparison with the corresponding case without the mass. However, in the analyzed ratio M/M_0 change it does not exceed the following values: 5%—classical theory: 1%—Timoshenko-like theory; 0·05%—generalized Timoshenko theory. From the given examples it can be concluded that for a low stiffness transversal material the difference between results obtained using three-dimensional and classical theories increases with the increase of M and it practically does not change using the improved theories. Thus, the results obtained by using the improved Timoshenko type theories may be applied in all cases of the parameters λ_1, λ_2 M/M_0 changes as well as for plates made from different composite materials with more stiff properties.

One can now analyze how the attached masses influence higher frequencies of the isotropic plate. In Table 8 the frequencies ($2 \leqslant s \leqslant 9$, where s denotes a frequency sequence in a corresponding spectrum) obtained by using the classical theory, Timoshenko type theory (with and without rotary inertia effects) and a generalized Timoshenko model (with a rotary inertia effect) are given. For each s number two corresponding frequencies are given: higher values are obtained for $M = 0{\cdot}1\ M_0$, lower for $M = 0{\cdot}2M_0$. For the attached masses' values in the Table 9 higher frequencies for the first 10 spectra are given, in the increasing sequence obtained using the three-dimensional theory.

The results given in the tables allow some conclusions about higher free isotropic plate vibration frequencies with the concentrated mass attached at the point $x_1 = y_1 = 0{\cdot}476$. It seems that the attached mass does not influence a higher-frequency spectrum corresponding to bending (for approximate and three-dimensional theories). That influence is even smaller in the case related to thickness jump. For instance, with a change of a ratio M/M_0 in the interval 0·1–2·0 the higher frequencies related to bending decrease by less than 0·4%. The added mass does not influence free vibration frequencies, the frequencies of the spectrum related to the rotation (IIA), because for that case $w = 0$. The three-dimensional theory shows no influence on the other spectra (not outlined by the two-dimensional theories) and (for example) corresponding to the symmetric vibration modes of an unloaded plate.

An influence of the attached mass on the higher vibration frequencies of the transversally isotropic plate ($G'/E = 0{\cdot}01$) with one concentrated mass are shown in Tables 10 (two-dimensional theories) and Table 11 (three-dimensional theory). The changes intervals of the geometric parameters λ_1, λ_2 and M/M_0 are the same as in the previously considered cases.

The analysis of the results given in the tables leads to a conclusion that an occurrence of the concentrated mass causes a frequency decrease (to 14%) for different modes related to bending (for improved theories and the three-dimensional theory). The results prove the lack of influence of the attached mass on higher frequencies of the spectra IIA and IIIA (using Timoshenko type theory and the generalized Timoshenko model with rotary inertia effects).

TABLE 8

High frequencies of the isotropic plate obtained by using Kirchhoff, Timoshenko and generalized Timoshenko models ($G/E = 0{\cdot}384$; $\lambda_1 = 5$, $\lambda_2 = 5$; $\nu = 0{\cdot}3$)

S	Kirchhoff theory	Timoshenko model				Generalized Timoshenko model		
		Without inertia		With inertia				
1	2	3	4	5	6	7	8	9
2	0·95994	0·79160	0·76236	3·7382	4·3615	0·74539	3·0950	3·6422
	0·05174	0·78294	0·75454	3·7382	4·3614	0·73729		3·6419
3	1·5409	1·1640	1·1144	3·8934	4·7895	1·0844	3·24045	4·0184
	1·3902	0·99358	0·96207	3·8934	4·7895	0·92746	3·2445	4·0184
4	1·6945	1·2104	1·1686	3·9935	5·0076	1·1299	3·3404	4·2228
	1·5416	1·1645	1·1148	3·9935	4·9975	1·0848	3·3404	4·2087
5	2·4987	1·6763	1·6015	4·1391	5·4046	1·5523	3·4793	4·5528
	2·4933	1·6727	1·5981	4·1391	5·4040	1·5488	3·4793	4·5526
6	3·1372	1·9335	1·8553	4·3257	5·8340	1·7902	3·6564	4·9256
	2·9608	1·8488	1·7699	4·3257	5·8240	1·7085	3·6564	4·9238
7	3·3085	2·0432	2·9537	4·3711	5·9026	1·8904	3·6993	4·9997
	3·2985	2·0409	1·9511	4·3711	5·8815	1·8882	3·6993	4·9898
8	3·8523	2·2701	2·1725	4·4605	6·1428	2·1010	3·7837	5·1861
	3·8522	2·2700	2·1724	4·4605	6·1427	2·1010	3·7837	5·1861
9	4·7866	2·6172	2·5105	4·6761	6·6065	2·4257	3·9869	5·5838
	4·7711	2·6112	2·5042	4·6761	6·6040	2·4196	3·9869	5·5832

TABLE 9

High frequencies of the isotropic plate using three-dimensional theory ($G/E = 0{\cdot}384$; $\lambda_1 = 5$, $\lambda_2 = 5$; $v = 0{\cdot}3$)

Three-dimensional theory								
1	2	3	4	5	6	7	8	9
	0·74869	0·88858	1·0886	1·1334	1·4050	1·4857	1·5558	1·7772
	0·73891	0·88858	0·8985	1·0890	1·4050	1·4792	1·5518	1·7772
1·7817	1·8907	1·9869	2·0998	2·6554	2·3314	2·5906	2·6657	2·8099
1·6751	1·8884	2·9869	2·0997	2·6554	2·3308	2·5906	2·6657	2·8099
2·9668	2·9847	3·2648	3·4246	3·4414	3·5820	3·6094	3·6653	3·7172
2·7151	2·9071	3·2648	3·3554	3·4414	3·5762	3·6094	3·6103	3·7172
3·8732	3·9716	4·0040	4·0720	4·1202	4·1785	4·2149	4·3710	4·4017
3·8732	3·9710	4·0040	4·0720	4·1202	4·1784	4·2149	4·3068	4·4014
4·7949	4·9632	5·3455	5·4470	5·4635	5·4798	5·5376	5·5731	5·6024
4·7545	4·9623	5·3454	5·4469	5·4635	5·4798	5·5364	5·5731	5·6024
5·7493	5·7817	5·9345	5·3457	6·4384	6·5217	6·5297	6·5899	6·6791
5·7487	5·7801	5·9345	5·3457	6·4384	6·4759	6·5297	6·5899	6·6791
6·7963	6·8253	6·8829	7·1706	7·3410	7·5180	7·9767	8·1133	8·3170
6·7963	6·8253	6·8829	7·1635	7·2852	7·5180	7·9746	8·0875	8·3167
8·5323	9·3196	9·4184	9·4258	9·4315	9·4747	9·5067	9·5289	9·5741
8·5323	9·2744	9·4184	9·4252	9·4315	9·4745	9·5025	9·5287	9·5741

TABLE 10

Influence of the attached mass of the higher frequencies of the transversal isotropic plate $G'/E = 0{\cdot}01$; $\lambda_1 = 5$, $\lambda_2 = 5$; $\nu = 0{\cdot}3$)

S	Kirchhoff's theory	Timoshenko's model				Generalized Timoshenko's model		
		Without inertia			With inertia			
2	0·95994	0·21966	0·21961	1·5119	2·4402	0·20155	1·3368	2·1710
	0·95174	0·21608	0·21602	1·5119	2·4402	0·19826	1·3368	2·1710
3	1·5409	0·27620	0·27616	1·8629	3·0558	0·25291	1·6537	2·7231
	1·3902	0·23802	0·23795	1·8629	3·0558	0·21819	1·6537	2·7231
4	1·6945	0·28183	0·28178	2·0639	3·4050	0·25820	1·8347	3·0363
	1·5416	0·28175	0·28171	2·0639	3·4049	0·25813	1·8347	3·0363
5	2·4987	0·36072	0·36068	2·3333	3·8702	0·33015	2·0769	4·4523
	2·4933	0·36012	0·36008	2·3333	3·8702	0·32960	2·0769	4·4523
6	3·1372	0·39491	0·39487	2·6502	4·4148	0·36130	2·3616	3·9397
	2·9605	0·38336	0·38333	2·6502	4·4147	0·35080	2·3616	3·9397
7	3·3085	0·41591	0·41588	2·7236	4·5407	0·38057	2·4275	4·0529
	3·2985	0·41568	0·41565	2·7236	4·5406	0·38031	2·4275	4·0529
8	3·8523	0·44995	0·44992	2·8649	4·7827	0·41159	2·5543	4·2688
	3·8522	0·44995	0·44992	2·8649	4·7827	0·41158	2·5543	4·2688
9	4·78666	0·50143	0·50140	3·1909	5·3398	0·45857	2·8467	4·7672
	4·77117	0·50058	0·50055	3·1909	5·3398	0·45779	2·8467	4·7672

TABLE 11

Influence of the attached mass of the higher frequencies of the transversely isotropic plate obtained by using three-dimensional theory (frequencies spectrum related to bending) ($G'/E = 0{\cdot}01$; $\lambda_1 = 5$, $\lambda_2 = 5$; $\nu = 0{\cdot}3$)

2	3	4	5	6	7	8	9
0·20977	0·26613	0·27083	0·34883	0·38294	0·40357	0·43739	0·48858
0·20640	0·22827	0·27074	0·34822	0·37122	0·40334	0·43738	0·48771

7. GENERAL CONCLUSIONS

1. The generalized Timoshenko type model and Timoshenko's model with the inertia rotary effects allow one to obtain practically exact results of frequency spectrum calculations related to bending in the considered geometrical ($5 \leqslant \lambda_1$, $\lambda_2 \leqslant 100$) and physical ($0{\cdot}01 \leqslant G'/E \leqslant 0{\cdot}38$) parameters ranges of isotropic and transversally isotropic plates.
2. Presence of a concentrated mass decreases the vibration frequencies corresponding to the analyses of unloaded plates by using the improved and three-dimensional theories as well as the Timoshenko type theory.
3. For high-frequency spectra of the isotropic and transversally isotropic plates the attached mass influence may be omitted (except for the bending process).

ACKNOWLEDGMENT

This work was supported by the Polish State Committee for Scientific Research (grant No. 8T11F01416)

REFERENCES

1. I. A. KILTCHEVSKI 1963 *Fundamentals of Shells Analytical Mechanics*. Kijew: Naukova Dumka (in Russian).
2. E. I. GRIGOLIUK, I. T. SIELIEZOV 1973 *Nonclassical Theories of Beams, Plates and Shells*. (Achievements of Science and Technics. Mechanics of hard deformable bodies. Moscow: WINITI (in Russian).
3. S. SRINIVAS and A. K. RAO 1971. *J. Franklin Institute* **291**, 469–481. A three-dimensional solution for plates and laminates.
4. S. SRINIVAS, C. V. J. RAO and A. K. RAO 1970 *Transactions of ASME E37* **3**, 868–870. Some results from an exact analysis of thick laminates in vibrations and buckling.
5. L. A. AINOLA and U. K. INGUL 1965 *Izvestia AN Est. SSR, Ser. fiz-mat. i techn. n* **14**, 3–63. Wave processes of plates and shells deformations (in Russian).
6. U. K. NIGUL 1963 *Izv. AN Est. SSR. Ser. fiz-mat. i tiechn.* 146–155. On application of the Lurie symbolic method in a three dimensional plate dynamical theory (in Russian)..
7. P. A. ŽILIN and T. P. ILICIEVA 1980 *Izvestia AN SSSR Miechanika Tvierdovo Tiela* 94–103. Vibration spectra and modes of a rectangular cubicoid on a basis of a three dimensional elastic and plates theories (in Russian).

8. V. I. Moskalienko 1968 *Izvestia AN Arm. SSR. Miechanika* **21**, 57–64. Free vibrations of thick plates (in Russian).
9. P. A. Žilin 1979 *Calculations of Space Constructions* Moscow: Nauka. A new approach for development of a new linear theory of thin elastic shells. **19**, (in Russian).
10. S. Srinivas, C. V. J. Rao and A. K. Rao 1970 *Journal of Sound and Vibration* **12**, 187–199. An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates.
11. R. D. Mindlin 1951 *Journal of Applied Mechanics*. Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates.
12. A. S. Boženov and S. K. Elmuratov 1978 *Stroitelna Miekhanika, Karaganda* 121–125. Influence of concentrated masses on forced vibrations of the orthotropic rectangular plates (in Russian).
13. V. A. Zarutskij 1982 *Prikladnaja Miekhanika* **18**, 50–56. Forced vibrations of a longitudinally supported cylindrical shell with locally jointed mass (in Russian).
14. A. P. Muchojed and N. I. Muchojed 1981 *Calculation and Applied Mathematics Kijów* 44, 62–65. On excited vibrations of orthotropic circled plate with a stiff additive, in Russian.
15. S. V. Kozlov 1981 *Doklady AN USSR. Series A* 47–51. On the parametric instability zones of an orthotropic cylindrical shell with the added masses (in Russian).
16. P. A. Laura and R. H. Gutierrez 1977 *Journal of Sound and Vibration* **55**, 49–53. Vibration of simply-supported plates of arbitrary shape carrying concentrated masses and subjected to a hydrostatic state of in plane stresses.
17. V. P. Cirkov 1973 *Theory of Shells and Plates*, 579–584, Moscow: Nauka, A change of stochastic vibration field caused by vibration pick-ups. (in Russian).
18. A. I. Lichodied 1973 *Izvestia AN SSSR. Miechanika Tvierdovo Tiela* 141–146. Nonlinear vibrations of shells with the concentrated masses (in Russian).
19. B. M. Karmaker 1978 *International Journal of Non-linear Mechanics* **13**, 451–459. Amplitude frequency characteristics of non-linear vibrations of a clamped elliptic plates carrying a concentrated mass.
20. P. S. Kovalczuk, S. V. Kozlov and A. N. Nosaèenko 1980 *Doklady AN USSR. Series* A, 43–47. On the problem of a cylindrical shell with an added mass dynamical instability (in Russian).
21. V. G. Palamartchuk 1980 *Prikladnaja Miekhanika* **16**, 22–27. Dynamical instability of a ribbed cylindrical shell with a mass jointed by springs (in Russian).
22. E. L. Makarovskij 1978 *Mathematical Methods of Cybernetics*, 35–39 Kijów: Naukova Dumka. On location load optimisation on the thin plates using a fundamental frequency criterion (in Russian).
23. J. G. Stojan and E. L. Makarovskij 1980 *Prikladnaja Miekhanika* **16**, 70–74. On a frequency rotational distribution of concentrated masses on thin plates (in Russian).
24. A. A. Fomin and A. S. Christienko 1971 *Nikolajev Ship Construction Institute Press* **50**, 71–75. Universal program of free vibration frequency determination of a system: plate-mass-spring (in Russian).
25. J. Awrejcewicz and V. A. Krysko Journal of Technical Physics. On the vibrations of orthotropic conical shells with added masses.